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Two problems in arithmetic geometry.
Explicit Manin-Mumford, and arithmetic Bernštein-Kušnirenko.

Présentée et soutenue par
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[2]Fou prend Tour f4, échec et mat.

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Contents

Introduction

The subject of this thesis lies in the field of arithmetic geometry, with a view towards toric geometry. We revisit geometric and arithmetic intersection theory to give computations on the closely related concepts of *torsion* and *height* of specific varieties.

This thesis consists of two independent chapters, the first one dedicated to the study of torsion in subvarieties of the torus and Abelian varieties, whereas the second one studies heights of 0-cycles of toric varieties.

The starting point of the first part of this thesis is the following question posed independently by Manin and Mumford, as stated by Lang in [48]: *If a curve in its Jacobian contains infinitely many points of finite period, is the curve of genus* 1*?* Motivated by this question, Lang states in [49, p. 220] the Manin-Mumford conjecture under the following form:

> *Let G be a torus or an Abelian variety in characteristic* 0*. Let V be a subvariety of G containing an infinite number of torsion points of G. Then V contains a finite number of translations of group subvarieties of G which contain all but a finite number of the torsion points in V.*

Here Lang refers as torus to the complex multiplicative group $\mathbb{G}_{\mathrm{m}}^n = (\mathbb{C}^\times)^n$ with the coordinatewise multiplication as its group action. Hence, torsion points are simply n-tuples of roots of unity.

We can replace the group subvarieties in the statement of the conjecture by *torsion cosets* of G, that is, irreducible algebraic subgroups of G translated by torsion points. So torsion points are torsion cosets by taking the trivial subgroup, and Manin-Mumford's conjecture can be reformulated as the statement that the Zariski closure of the torsion points in V is a finite union of torsion cosets.

For the case when G is a torus, the conjecture was first proved by Ihara, Serre and Tate [48] when V is a curve, and by Laurent [52] for any variety, although it could be already deduced from previous results of Mann [57]. The Abelian counterpart of this conjecture was proven by Raynaud [68, 69]. Furthermore, Hindry [42] also proved that the conjecture holds when G is replaced by any algebraic commutative group.

Since Manin-Mumford's conjecture has been proved, part of the focus of interest has shifted to bounding (explicitly and effectively) the number and degree of the torsion cosets in the variety V. To be more precise, ordering torsion cosets by inclusion yields a notion of maximality of torsion cosets that are contained in V; the aim is to obtain a bound on the number and the degree of maximal torsion cosets. We denote by $\overline{V_{\mathrm{tors}}}$ the Zariski closure of the torsion points. From here on forward, we present both the toric and abelian instances of Manin-Mumford's conjecture separately. More information and precisions are given in §1.1.

Let us first restrict ourselves to the toric setting of the conjecture, and give an extensive overview of the results in this case. Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a variety defined over a number field $\mathbb{K}$ by polynomials of degree at most δ, and height at most η. In this case, Laurent's theorem gives a bound for the number of torsion cosets in V in terms of n, δ, η and the degree $[\mathbb{K} : \mathbb{Q}]$. But his result is not effective, as he actually proves a particular case of the Mordell-Lang conjecture. Later, Bombieri and Zannier [9] showed that both the number of maximal torsion cosets and their degree can be bounded solely in terms of n and δ. Both parameters are needed, since we can build a simple example to show that the bound must depend on both the dimension of the ambient space and the degree of the variety as follows. If V is the hypersurface of degree δ defined as the zeroes of the polynomial

$$f(x_1, \ldots, x_n) = n - x_1^\delta - \cdots - x_n^\delta \in \mathbb{Q}[x_1, \ldots, x_n];$$

then it is easy to check that the only torsion points lying on V are n-tuples of δ-th roots of unity. Hence the number of maximal torsion cosets in V equals its number of torsion points, which amounts to δ^n.

Simultaneously to the results of Bombieri and Zannier, Schlickewei [76] gave an upper bound for the number of solutions in roots of unity of a linear equation that depends only on the number of variables. This result was then used by Schmidt [77] to give an effective bound of the number of maximal torsion cosets of a variety V in terms of n and δ. Further contributions in this direction where obtained by the improvement of Schlikewei's result done by Evertse in [31].

Much sharper bounds follow from the study of the (logarithmic) Weil height of points in the torus $\mathbb{G}_{\mathrm{m}}^n$. Since torsion points are the points of Weil height zero, the results on points of *sufficiently* small height can be used to deduce bounds on the number of maximal torsion cosets. The results in this direction by David and Philippon [28], Rémond [70], and Amoroso and Viada [2], and allow to obtain a bound on the number of maximal torsion cosets in V which is polynomial in δ.

From an algorithmic point of view, the first steps towards finding the solutions in roots of unity where provided by Mann [57] and Conway-Jones [26]. Their work on relations between roots of unity precedes the formulation of Manin-Mumford's conjecture by Lang, and further motivates the study of torsion points in the toric case. A first algorithm on finding the torsion cosets of a general variety in the torus, is given by Sarnak and Adams [74]. More recent developments on relations of roots of unity by Dvornicich and Zannier [30] also improve the existing bounds in this direction.

In [73], Ruppert considers the case of a non-torsion irreducible curve C embedded in $(\mathbb{P}^1)^n$ of multidegree $(d_1, \dots, d_n)$, $d_i > 0$ for all i. He gives an algorithm to find the torsion points in C, which bounds its number by $22 \min(d_i) \max(d_i)$. His algorithm, however doesn't extend to higher dimensional varieties except for a small family of surfaces. Nevertheless, by a further study of the higher dimensional case, he provides a way of deducing bounds on the number of positive dimensional maximal torsion cosets in V, from a bound on its isolated torsion points (they correspond to maximal torsion cosets of dimension 0). These results together with some explicit examples motivate him to formulate the following conjecture:

Conjecture (Ruppert)**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial of multidegree $(d_1, \dots, d_n)$, $d_i > 0$ for all i, and let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be the variety defined by f. The number of isolated torsion points in V can be bounded above by $c_n\, d_1 \cdots d_n$, where c_n is an effective constant depending only on n.*

With the above mentioned study by Ruppert, an affirmative answer to this conjecture would imply that for a variety V defined by polynomials of degree δ, one can give a bound on the number of maximal torsion cosets which is polynomial in δ and of degree n.

Beukers and Smyth [5] reconsider this problem for curves in $\mathbb{G}_{\mathrm{m}}^2$, giving a refinement of Ruppert's bound for curves defined by sparse polynomials. Given $f \in \mathbb{C}[x, y]$ they provide a family of polynomials which are closely related to f, such that the solutions in roots of unity of f are also solutions of one of the polynomials in this family. They then use Berštein-Kušnirenko's theorem to give a bound in terms of the Newton polytope of f. More concretely, if $\Delta = \mathrm{conv}(\mathrm{supp}(f))$ is the Newton polytope of f, that is the convex hull in $\mathbb{R}^n$ of the exponents appearing in the monomial expansion of f, and the curve defined by the zeroes of f is non-torsion, then it contains at most $22\, \mathrm{vol}_2(\Delta)$ torsion points, where vol_2 represents the volume associated to the Lebesgue measure on $\mathbb{R}^2$.

Later Aliev and Smyth generalized this strategy to higher dimensional varieties in [1]. They did that by using projections and resultants which yields a bound which is exponential in δ. However, the result they obtained is distant from their original objective, which was to prove the following stronger version of Ruppert's conjecture that takes into account the sparsity as Beukers and Smyth do in [5].

Conjecture (Aliev-Smyth)**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial, $\Delta = \mathrm{conv}(\mathrm{supp}(f))$ be its Newton polytope, and $V \subset \mathbb{G}_{\mathrm{m}}^n$ be the hypersurface defined by f. The number of isolated torsion points in V can be bounded above by $c_n\, \mathrm{vol}_n(\Delta)$, where c_n is an effective constant depending only on n, and vol_n is the volume associated to the Lebesgue measure on $\mathbb{R}^n$.*

It is easy to see that this conjecture implies the conjecture of Ruppert: If f is of multidegree $(d_1, \dots, d_n)$, $d_i > 0$, then we have that the support of f lies in the box $\prod_{i=1}^n [0, d_i]$, implying that $\mathrm{vol}_n(\Delta) \leq \mathrm{vol}_n \left(\prod_{i=1}^n [0, d_i]\right) = d_1 \cdots d_n$.

In the first part of Chapter 1 it is our purpose to prove both of these conjectures. The strategy can be divided in four steps:

1. an extension of the argument for plane curves of Beukers and Smyth [5] to varieties of any dimension in $\mathbb{G}_\mathrm{m}^n$;

2. an interpolation argument using upper and lower bounds on the Hilbert function in a similar fashion to Amoroso and Viada [2];

3. an application of the two induction techniques of Viada in [2] to replace straightforward intersection by Bézout's theorem (this gives a first bound in terms of the usual degree);

4. an implementation of a result on ellipsoids in metric spaces of John [44] to translate the previous result to a notion of degree associated to convex polytopes and prove the conjectures.

For the first step, let us assume that $V \subset \mathbb{G}_\mathrm{m}^n$ is an irreducible variety of positive dimension (incompletely) defined by polynomials of degree at most δ. We give a geometric analogue to Beuker and Smyth's results in [5] that applies to V, and thereby construct a variety V' defined by polynomials of degree δ up to multiplication by a constant depending only on n. Moreover, this variety satisfies that $\overline{V_\mathrm{tors}} \subset V \cap V' \subsetneq V$ (Lemma 1.2.5 and Proposition 1.2.6).

In the second step, we use the upper and lower bounds on the Hilbert fuction, results of Chardin [23], and Chardin and Philippon [24] respectively, to prove the existence of a hypersurface Z that plays a similar role as the variety V' obtained in the first step. More concretely, in Theorem 1.2.16, we prove that there is a hypersurface Z such that $\overline{V_\mathrm{tors}} \subset V \cap Z \subsetneq V$, and has degree δ up to a multiplicative factor depending only on n.

In the third step we intersect inductively with hypersurfaces as the ones mentioned above. To avoid an exponential growth of the exponent of δ from a such iterative process, we use Amoroso and Viada's approach in [2]. These techniques yield our first main result, Theorem 1.2.18.

Theorem A. *Let $V \subset \mathbb{G}_\mathrm{m}^n$ be a variety of dimension d, defined by polynomials of degree at most δ. Let V_{tors}^j be the union of the irreducible components of $\overline{V_{tors}}$ of dimension j, $j = 0, \dots, d$. Then*

$$\deg(V_{tors}^j) \leq c(n)\, \delta^{n-j},$$

for every $j = 0, \dots, n$, where $c(n)$ is an effective constant that only depends on n.

From this result, one can readily deduce Ruppert's conjecture via algebraic group homomorphisms (Corollary 1.2.19). However, we need an extra tool to prove Aliev-Smyth's conjecture.

For the last step, let us introduce the notion of degree related to a convex polytope $\Delta \subset \mathbb{R}^n$ with integer vertices. Given a variety $W \subset \mathbb{G}_{\mathrm{m}}^n$ of dimension d, we define $\deg_\Delta(W) = \mathrm{card}(W \cap Z)$ where Z is a variety of codimension d defined by d generic polynomials with Newton polytope Δ (Definition 1.2.22). Then, by means on a result of John [44], we obtain our second main result, Theorem 1.2.23.

Theorem B. *Let $\Delta \subset \mathbb{R}^n$ be a convex polytope with integer vertices. Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a variety of dimension d, defined by polynomials with Newton polytope contained in Δ. Then*

$$\deg_\Delta(V^j_{tors}) \leq \widetilde{c}(n)\ \mathrm{vol}_n(\Delta)$$

for every $j = 0, \ldots, d$, where $\widetilde{c}(n)$ is an effective constant that only depends on n.

From this statement we readily deduce Aliev-Smyth's conjecture by taking Δ to be exactly the Newton polytope of f, and $j = 0$.

Let us now turn to the case when $G = A$ is a complex Abelian variety. Fix $\iota\colon A \hookrightarrow \mathbb{P}^n$ a closed immersion into a projective space of some dimension n, and identify any subvariety $X \subset A$ with its image by ι. One then considers the degree of X as the usual degree in $\mathbb{P}^n$. In the sequel, when something is said to depend on A, it may also depend implicitly on the choice of ι.

Mainly because of the more intricate structure of torsion points of A, explicit bounds on the Manin-Mumford conjecture are less common than their toric counterparts. One should nevertheless emphasize that the particular case of a curve C embedded in its Jacobian has given rise to explicit and effective bounds on the number of torsion points in C. We highlight the results of Coleman [25] using p-adic integration, and of Buium [15] relying on p-jets.

For the general case; given $V \subset A$, Hindry's proof of Manin-Mumford's conjecture in [42] yields a bound on the number of maximal torsion cosets in V which is effective up to a constant depending on Galois representations. However these bounds can hardly be made explicit as discussed in [41]. Further studies of Bombieri and Zannier [10] on the Néron-Tate height show that it is possible to give a bound just in terms of the degree of V, and data coming from A. By means of model-theoretic methods, Hrushovski [43] gives an explicit geometric bound on the Manin-Mumford conjecture whose dependence on $\deg(V)$ is doubly exponential in parameters coming from A.

Given the result obtained for the toric Manin-Mumford conjecture (Theorem A), one expects a much better dependence on the degree of V. More concretely, say $\dim(A) = g$ and V is defined in $\mathbb{P}^n$ by the intersection of hypersurfaces of degree at most δ, then one might expect to bound the number of maximal torsion cosets in V by $c(A)\,\delta^g$, where $c(A)$ is a constant only depending on A.

In the second part of Chapter 1, we focus on obtaining a such bound when A is defined over $\overline{\mathbb{Q}}$. The strategy follows a similar structure to the one listed above in the toric case, and can be divided in three steps:

1. a study of the Galois action on the torsion of A to extract geometric information on torsion points, from which we are able to deduce a bound on the number of torsion points in the case when the variety V is a curve;
2. an interpolation argument using upper and lower bounds on Hilbert functions relative to the inclusion $V \subset A$;
3. an application of the two induction techniques of Viada in the abelian setting, from which we obtain the expected bound.

For the first step, let us assume that K is a "big enough" finite extension of the field of definition of A. A result of Bogomolov [7], later improved by Serre [80], states that there exists a constant $\mathrm{c} \in \mathbb{N}_{>0}$, which is not known to be effectively computable, such that for every point $P \in A$ of finite order, and every positive integer k prime to the order of P, there exists an automorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that

$$P^\sigma = \overbrace{P + \cdots + P}^{k^{\mathrm{c}} \text{ times}}.$$

By means of this result, we are able to give an explicit construction of a variety $V' \subset A$ such that $\overline{V_{\mathrm{tors}}} \subset V \cap V' \subsetneq V$ (Propositions 1.3.4, 1.3.7, and 1.3.8). Moreover, the degree of V' can be expressed as the degree of V up to an explicit multiplying factor depending on $g = \dim(A)$ and c. This allows us to give a preliminary bound in the case when V is a curve (Proposition 1.3.9 and the subsequent remark). We derive the following result, which can be seen as the abelian analogue to Beukers-Smyth's bound.

Proposition. *Let $C \subset A$ be an irreducible algebraic curve of genus greater than* 1*. Then*

$$\#\, C_{\mathrm{tors}} \leq (2^{4g+\mathrm{c}}\mathrm{c}^{2g} + 2^{2g+1} - 1)\,\deg(C)^2.$$

In the second step we make use of the upper and lower bounds on the Hilbert function relative to the homogeneous coordinate ring of $\mathbb{P}^n$, due to Chardin [23], and Chardin and Philippon [24], respectively. Assume that V is defined in $\mathbb{P}^n$ by the intersection of hypersurfaces of degree at most δ. Then, from V', we derive in Proposition 1.3.13 a hypersurface $Z \subset \mathbb{P}^n$ of degree δ up to a multiplying factor depending on A and c, such that $\overline{V_{\mathrm{tors}}} \subset V \cap Z \subsetneq V$.

The last step consists of applying the same double induction as we use to prove Theorem A. From it we obtain the following explicit bound for the abelian Manin-Mumford conjecture, Theorem 1.3.14.

Theorem C. *Let $V \subset A$ be a subvariety of dimension d, defined in $\mathbb{P}^n$ as the intersection of hypersurfaces of degree at most δ. Let V^j_{tors} denote the union of the irreducible components of $\overline{V_{tors}}$ of dimension j, $j = 0, \ldots, d$. Then*

$$\deg(V^j_{tors}) \leq c(A)\delta^{g-j},$$

for every $j = 0, \ldots, n$, where $c(A)$ is an explicit constant only depending on the dimension of A, n, $\deg(A)$, and the constant c.

The bound on the number of maximal torsion cosets in V given by this theorem is effective, up to the non-effective constant c. This constant, however, was conjectured by Lang to equal 1 for points of order high enough, and any effective result on the computation of c will automatically yield our constant effective.

On the second part of this thesis we focus on the arithmetic of toric varieties. The foundations for the study of toric varieties were laid down in the 1970's by independent work of Demazure [29], Kempf,Knudsen,Mumford and Saint-Donat [45], Miyake and Oda [64], and Satake [75]. Fixed a field $\mathbb{K}$, a *toric variety* can be defined as algebraic variety X containing densely a *torus* or multiplicative group $(\mathbb{K}^\times)^n$, and such that the action of $(\mathbb{K}^\times)^n$ on itself by translations extends to X. There is a one-to-one correspondence between toric varieties and *fans*, which enables an extensive and deeply developed dictionary between the algebraic geometric properties of toric varieties and the convex geometric properties of fans and polytopes.

An interesting example where these relations prove to be useful, which is also the main motivation for this part, is Bernštein-Kušnirenko's theorem [4, 47]. This theorem gives a bound on the number of isolated zeros of a system of Laurent polynomials over $\mathbb{K}$, in terms of the mixed volume of their Newton polytopes. It follows from the one-to-one correspondence between convex polytopes in $\mathbb{R}^n$ with integer vertices, and toric varieties endowed with a line bundle that is invariant by the torus action, and the properties implied by this bijection. For n Laurent polynomials $f_1, \ldots, f_n \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with respective Newton polytopes $\Delta_1, \ldots, \Delta_n$, the number of isolated solutions of the system of equations $f_1 = \cdots = f_n = 0$ in $(\overline{\mathbb{K}}^\times)^n$ is bounded by the mixed volume $\mathrm{MV}_n(\Delta_1, \ldots, \Delta_n)$ associated to the Lebesgue measure of $\mathbb{R}^n$ (Definiton 2.2.7). Moreover, this is an equality for a generic choice of polynomials. In comparison with the classical Theorem of Bézout, it does not only take into account the degree of the polynomials, but the distribution of all exponents appearing in the monomial expansions. Thus it is a refinement of Bézout's theorem that allows to predict when a system of equations has a small number of solutions in the torus. As an illustrative example, let $d, H \in \mathbb{N}_{>0}$ and consider the system defined by the following Laurent polynomials

$$f_i = x_i - Hx_{i-1}^d \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \quad i = 1, \ldots, n. \tag{1}$$

After an easy computation, one has that Bernštein-Kušnirenko's bound on the number of solutions in $(\overline{\mathbb{K}}^{\times})^n$ of this system of polynomials is 1, and indeed $(H, \dots, H^{1+d+\cdots+d^{n-1}})$ is this system's only zero in the torus. Notice that is much smaller than the product of their degrees d^n.

Bernštein-Kušnirenko's theorem has had a considerable impact since its formulation. As it provides a simpler way for dealing with polynomial systems of polynomial equations, it has seen many applications on this regard, for example in computational algebra [36,83]. Furthermore it has also contributed the other way around, providing for instance a proof of the Alexandrov-Fenchel inequality (for which a direct approach in convex geometry is rather difficult) by algebraic means via the Hodge inequality, see [84] and Addedum 3 by Khovanskiĭ in [16]. Because of its relevance, it has also inspired a great number of generalizations, a brief discussion on this matter can be found in [83, Chapter 3]. We point out the refinement of Philippon and Sombra [67] which gives a bound in terms of a mixed integral of convex functions, and serves as first precursor of some of the work considered below.

When $\mathbb{K}$ is endowed with an arithmetic structure, it is also of interest to have a control on the *height* or *complexity* of the solution set of a such family of Laurent polynomials. The notion of height of a point was first developed by Siegel, Northcott and Weil among others as a way of measuring the "size" of a point, and is an essential tool in diophantine geometry. In higher dimension, this concept extends as an analogue of the degree of a variety that measures the complexity of the representation of it, for example via its Chow form. Therefore, it is also of relevance in algebraic geometry and effective computational algebra, for instance when dealing with effective versions of the Nullstellensatz [27,38,46]. This further motivates an arithmetic Bernštein-Kušnirenko type bound.

For simplicity of exposition, let us consider $\mathbb{K} = \mathbb{Q}$ although the results exposed below also hold for the more general setting of adelic fields satisfying the product formula. The usual height of a point in $(\mathbb{Q}^{\times})^n$ is the Weil height, which is defined for each $(x_1, \dots, x_n) \in (\mathbb{Q}^{\times})^n$ as

$$\mathrm{h_W}(x_1, \dots, x_n) = \sum_{p \in \{\text{primes}\} \cup \{\infty\}} \log \max\{1, |x_1|_p, \dots, |x_n|_p\},$$

where $|\cdot|_\infty$ and $|\cdot|_p$, respectively represent the absolute value and p-adic absolute value normalized in the usual way. However, the general definition of height is richer than just considering the Weil height and allows a wider consideration of alternatives heights. For example, one can define a height attached to a monomial map $\varphi : (\mathbb{Q}^{\times})^n \to (\mathbb{Q}^{\times})^r$ by taking the inverse image of the Weil height in $(\mathbb{Q}^{\times})^r$; that is, for every $\boldsymbol{x} \in (\mathbb{Q}^{\times})^n$, we define its height associated to φ as $\mathrm{h}_{\varphi^*\mathrm{W}}(\boldsymbol{x}) = \mathrm{h_W}(\varphi(\boldsymbol{x}))$.

To give an example in which the difference between distinct considerations of heights is emphasized, let us come back to the system of polynomial equations defined by (1), for $d, H \in \mathbb{N}_{>0}$. As mentioned above, the zero set defined by these polynomials consists of a simple point $\boldsymbol{p} = (H, \dots, H^{1+d+\cdots+d^{n-1}})$. Its Weil height $\mathrm{h}_{\mathrm{W}}(\boldsymbol{p}) = \sum_{i=1}^{n} d^{i-1} \log H$ grows polynomially with the degrees of the polynomials. On the contrary, by considering the height attached to the monomial map $\varphi : (\mathbb{Q}^{\times})^n \to (\mathbb{Q}^{\times})^n$, defined by $(x_1, \dots, x_n) \mapsto (x_1, x_2x_1^{-d}, \dots, x_nx_{n-1}^{-d})$, we obtain $\mathrm{h}_{\varphi^*\mathrm{W}}(\boldsymbol{p}) = \log H$. One can interpret this phenomenon as the fact that the complexity of a point depends on the representation we use. The motivation behind an arithmetic version of Bernštein-Kušnirenko's bound is to give a way of predicting heights of zero sets of systems of Laurent polynomials in terms of the monomial structure of the polynomials and the given height function.

Arithmetic analogues of Bézout's theorem were proved using Arakelov geometry by Faltings [32] and Bost, Gillet and Soulé [11], although previous versions for heights that arise also without Arakelov theory were already known beforehand by Nesterenko [63], and Philippon [65]. As for Bernštein-Kušnirenko's theorem, a first result by Maillot [56] gives a bound for canonical heights associated to the toric divisors (which are generalizations of Weil heights for toric varieties), this result however is not completely effective. A further study in this direction was later done by Sombra [82].

In Chapter 2 we present an arithmetic Bernštein-Kušnirenko bound which improves the previous results obtained in this direction, and generalize them to adelic fields satisfying the product formula and height functions associated to arbitrary nef toric metrized divisors. This chapter is divided into three parts, where the two initial ones serve mostly as an exposition of the objects that are fundamental in the third one for stating and proving the main theorem.

In the first part, we give a brief overview on the geometry of toric varieties, mainly describing the correspondence between toric divisors and their convex analogues, and their behaviour in intersection theory. The purpose here is to lay the geometric groundwork that is essential in the follow up. By doing so, we also present a proof of the classical Bernštein-Kušnirenko theorem. This defines the strategy we use in our subsequent proof of our arithmetic version of this theorem.

In the second part, we present the arithmetic objects that are the centrepiece of the sequel. We introduce the notion of adelic field, and detail a construction of adelic field extension that preserve the product formula. For normal projective varieties over adelic fields, we describe (global) heights of 0-cycles attached to metrized divisors. Afterwards, we extend this definition recursively, and give a well-defined notion of (global) height for general cycles with respect to metrized divisors which are generated by small sections. For such metrized divisors, this definition is an extension to adelic fields satisfying the product formula of the equivalent one for global fields in [19]. Most notably, under these assumptions, arithmetic intersection behaves similarly to its counterpart in algebraic geometry. When restricting to toric varieties, Burgos, Philippon and Sombra [19] have done a thorough study on the arithmetic of toric varieties, relating arakelovian properties with convex geometry, and exploring the implications of these relations. As such, their work is central to our study. Thus, we present their characterizations of (semipositive) metrized toric divisors $\overline{D}$ in terms of concave functions, *metric functions* $\{\psi_{\overline{D},v}\}$ and *roof functions* $\{\vartheta_{\overline{D},v}\}$ (Proposition 2.3.28), and the implications of these when dealing with their associated heights.

Finally, we prove our arithmetic Bernštein-Kušnirenko's bound. The following statements hold for general adelic fields; however, for simplicity, herein we present them in the case when our adelic field is $\mathbb{Q}$ with the usual set of absolute values as described above. The starting point is one of the principal results in [19], which identifies the height of a toric variety with respect to metrized toric divisors with a sum of mixed integrals of the corresponding roof functions. The key point of our proof is to associate to a Laurent polynomial f, a metrized toric divisor that is generated by small sections and such that the section given by f is small: for a given Laurent polynomial $f = \sum_{j=0}^{r} \alpha_j \boldsymbol{x}^{\boldsymbol{m}_j}$, where $\alpha_j \in \mathbb{Q}^\times$ and $\boldsymbol{m}_j \in \mathbb{Z}^n$ for every j, with Newton polytope $\Delta = \mathrm{conv}(\boldsymbol{m}_j)$, we define the concave functions $\vartheta_p : \Delta \to \mathbb{R}$, as

$$\vartheta_p(\boldsymbol{x}) = \begin{cases} \max\limits_{\boldsymbol{\lambda}} \Big(\sum\limits_{j=0}^{r} \lambda_j \log \dfrac{|\alpha_j|_p}{\lambda_j} \Big), & \text{for } p = \infty; \\[2ex] \max\limits_{\boldsymbol{\lambda}} \Big(\sum\limits_{j=0}^{r} \lambda_j \log |\alpha_j|_p \Big), & \text{for } p \text{ prime}; \end{cases} \tag{2}$$

the maximum being over all $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_r) \in \mathbb{R}^{r+1}_{\geq 0}$ such that $\sum_j \lambda_j = 1$ and $\sum_j \lambda_j m_j = x$. We then prove that the metrized toric divisor associated to these ϑ_p's is generated by small sections, and f is a small section of this divisor. The main result in this chapter, Theorem 2.4.5, states the following.

Theorem D. *Let X be a proper toric variety and $\overline{D}_0$ a nef toric metrized divisor on X with corresponding roof functions $\{\vartheta_{0,p} : \Delta_0 \to \mathbb{R}\}_p$. Let $f_1, \dots, f_n \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $\{\vartheta_{i,p} : \Delta_i \to \mathbb{R}\}_p$ be the roof functions associated to each f_i as in* (2)*. Then the height with respect to $\overline{D}_0$ of the 0-cycle defined by the system of f_i's is bounded by*

$$\mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n)) \leq \sum_{p \in \{\text{primes}\} \cup \{\infty\}} \mathrm{MI}(\vartheta_{0,p}, \dots, \vartheta_{n,p}).$$

We also give a second bound in terms of the mixed volumes of the Newton polytopes of the f_i's, and their logarithmic lengths, $\ell(f_i)$ (Definition 2.4.6). We readily derive from Theorem D, and basic properties of mixed integrals, that

$$\begin{aligned} \mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n)) \leq{}& \mathrm{MV}_M(\Delta_1, \dots, \Delta_n) \Big(\sum_p \max \vartheta_{0,p} \Big) \\ &+ \sum_{i=1}^{n} \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) \ell(f_i). \end{aligned}$$

This bound is easier to compute than the one of Theorem D, and in many cases already gives a good approximation to the actual height, as illustrated in Example 2.4.11. Nevertheless, we show the loss of precision of the bounds that occurs when passing from mixed integrals to mixed volumes (Example 2.4.12). We conclude by giving an application of these results to $\boldsymbol{u}$-resultants and rational univariate representation of 0-cycles.

Chapter 1

Explicit bounds on the Manin-Mumford conjecture

In this chapter we focus on effectiveness questions around the toric version of the Manin-Mumford's problem. The first half is devoted to the results proven in [58]. We give sharp bounds on the number of maximal torsion cosets in a subvariety of the complex algebraic torus, which prove the conjectures of Ruppert, and Aliev and Smyth on the number of isolated torsion points of a hypersurface. Furthermore, we present a work in progress in collaboration with Aurélien Galateau regarding analogous results for abelian varieties [35].

1.1 State of the art

1.1.1 The case of the torus

Let $\mathbb{G}_\mathrm{m}^n = (\mathbb{C}^\times)^n$ be the *multiplicative group* or *complex algebraic torus* of dimension n. We may identify $\mathbb{G}_\mathrm{m}^n$ with the Zariski open subset $x_1 \cdots x_n \neq 0$ in $\mathbb{A}^n_\mathbb{C}$, with the usual multiplication

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n).$$

In the following, a point $(x_1, \ldots, x_n) \in \mathbb{G}_\mathrm{m}^n$ is denoted by $\boldsymbol{x}$. In particular, $\mathbf{1} = (1, \ldots, 1)$ represents the identity element. Moreover, given any subset $S \subset \mathbb{G}_\mathrm{m}^n$ and a point $\boldsymbol{x} \in \mathbb{G}_\mathrm{m}^n$, we extend the operation above and denote by $\boldsymbol{x} \cdot S$ (or simply $\boldsymbol{x}S$) the *translation* of S by $\boldsymbol{x}$; that is $\boldsymbol{x} \cdot S = \{\boldsymbol{x} \cdot \boldsymbol{y} \mid \boldsymbol{y} \in S\}$.

A *torsion point* of $\mathbb{G}_\mathrm{m}^n$ is an n-tuple of roots of unity. We denote by

$$\mu_k = \{\zeta \in \mathbb{G}_\mathrm{m} \mid \zeta^k = 1\}$$

the subgroup of k-th roots of unity. Hence

$$\mu_k^n = (\mu_k)^n \quad \text{and} \quad \mu_\infty^n = \bigcup_{k\in\mathbb{N}_{>0}} \mu_k^n$$

represent, respectively, the subgroup of *k-torsion points* and the subgroup of *torsion points* of $\mathbb{G}_\mathrm{m}^n$. A *subtorus* $H \subset \mathbb{G}_\mathrm{m}^n$ is an irreducible algebraic subgroup of $\mathbb{G}_\mathrm{m}^n$. It is isomorphic (as an algebraic group) to $\mathbb{G}_\mathrm{m}^r$, for some $0 \leq r \leq n$, and the torsion points of $\mathbb{G}_\mathrm{m}^n$ are Zariski dense in any such subtorus. A *torsion coset* is a translate $\boldsymbol{\omega} \cdot H$ of a subtorus H by a torsion point $\boldsymbol{\omega} \in \mu_\infty^n$.

Let V be a subvariety of $\mathbb{G}_\mathrm{m}^n$, not necessarily irreducible, we denote by V_tors the set of torsion points contained in V, and we call its Zariski closure in $\mathbb{G}_\mathrm{m}^n$ the *torsion subvariety* of V:

$$\overline{V_\mathrm{tors}} = \overline{V \cap \mu_\infty^n}.$$

We say that a torsion coset $\boldsymbol{\omega} \subset V$ is maximal in V if it is maximal by inclusion.

Lang, inspired by a question that was posed to him by Manin and that arises, independently, from the work of Mumford, states in [48] what was to be known as the Manin-Mumford conjecture. For the moment we restrict ourself to the toric version of this (former) conjecture. This asserts that, if $V \subset \mathbb{G}_\mathrm{m}^n$ is an irreducible subvariety and V_tors is Zariski dense in V, then V is a torsion coset of $\mathbb{G}_\mathrm{m}^n$. In other words, the torsion subvariety of V is a union of torsion cosets of $\mathbb{G}_\mathrm{m}^n$. Lang gives proofs by Ihara, Serre, and Tate for the case when V is a curve in $\mathbb{G}_\mathrm{m}^2$, see *loc. cit.* and [49]. The proof for higher dimensional varieties follows, independently, from the work of Laurent [52, Théorème 2], and of Sarnak and Adams [74, Proposition 1.6].

In the sequel, we focus on finding a sharp upper bound for the number of maximal torsion cosets in V and their degrees. Assume that $V \subset \mathbb{G}_\mathrm{m}^n$ is defined over a number field $\mathbb{K}$ by a set of polynomials of degree at most δ and height at most η. As a consequence of the finiteness, Laurent's proof yields a bound for the number of maximal torsion cosets in V in terms of n, δ, η and $[\mathbb{K} : \mathbb{Q}]$. However, to obtain this bound, Laurent uses Schmidt's subspace theorem which is not effective. Bombieri and Zannier [9], following the work of Zhang [86], show that both the number of maximal torsion cosets in V and the their degree can be bounded just in terms of n and δ. Contemporarily, Schlickewei [76] gives an upper bound for the number of solutions in roots of unity of a linear equation (which depends only on the number of variables), and Schmidt [77] uses this result to give an alternative prove of the fact that the number of maximal torsion cosets in V can be bounded in terms of n and δ. By combining Schmidt's techniques with Evertse's improvement of Schlickewei's result in [31], we can bound the number of maximal torsion cosets in V by

$$(11\delta)^{n^2} \binom{n+\delta}{\delta}^{3\binom{n+\delta}{\delta}^2}.$$

Results of Mann [57], Conway and Jones [26] and, more recently, Dvornicich and Zannier [30] on the vanishing subsums of linear relations of roots of unity provide different algorithms for finding all the maximal torsion cosets in a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. The proof of Sarnak and Adams [74] of the toric Manin-Mumford conjecture, derives from a result of this type [74, Lemma 3.1], proposed to them by Cohen, and implies an algorithmic approach to this problem.

Furthermore, Ruppert [73] considers the problem of a non-torsion irreducible curve C in $\mathbb{G}_{\mathrm{m}}^n \hookrightarrow (\mathbb{P}^1)^n$ of multidegree $(d_1, \dots, d_n)$, $d_i > 0$ for all i, and obtains that the number of torsion points in C can be bounded above by

$$22 \min_i(d_i) \max_i(d_i).$$

In fact, he starts by treating the case of plane curves (so $n = 2$) and obtains the following sharper bound on the number of torsion points in C:

$$\#C_{\mathrm{tors}} \leq 22\, d_1 d_2 - 2\, d_1 - 2\, d_2.$$

In general, the approach of Ruppert doesn't extend to higher dimensional varieties, but after an extended study of them, he states the following conjecture:

Conjecture 1.1.1 (Ruppert)**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ have multidegree $(d_1, \dots, d_n)$, $d_i > 0$. The number of isolated torsion points on $Z(f) \subset \mathbb{G}_{\mathrm{m}}^n$ can be bounded above by $c_n d_1 \cdots d_n$, where c_n is a constant depending only on n.*

Beukers and Smyth [5] reconsider this problem for curves in $\mathbb{G}_{\mathrm{m}}^2$, refining this bound by giving one in terms of the volume of a Newton polytope of the curve. Given $f \in \mathbb{C}[x, y]$ a polynomial, they show that each torsion point in $Z(f)$, lies in the variety given by one of the following polynomials:

$$\begin{array}{ll} f_1(x,y) = f(-x,y), & f_4(x,y) = f(x^2,y^2), \\ f_2(x,y) = f(x,-y), & f_5(x,y) = f(-x^2,y^2), \\ f_3(x,y) = f(-x,-y), & f_6(x,y) = f(x^2,-y^2), \\ & f_7(x,y) = f(-x^2,-y^2). \end{array}$$

Recall that the *support* of a polynomial is the finite subset of $\mathbb{Z}^n$ given by the exponents of its monomials. Observe then that the supports of $f_1, \dots, f_3$ and $f_4, \dots, f_7$ are, respectively, the one of f and a dilation by 2 of the one of f. Then, by Bernštein-Kušnirenko's theorem (a toric analogue of Bézout's theorem, see Theorem 2.2.10), they obtain that the number of isolated torsion points of $Z(f)$ is bounded above by

$$22 \operatorname{vol}_2(\Delta); \tag{1.1.1}$$

where $\Delta = \operatorname{conv}(\operatorname{supp}(f))$ is the convex hull of the support of f, and vol_2 represents the volume associated to the Lebesgue measure on $\mathbb{R}^2$. As to fix notations, we precise that

Δ is called the *Newton polytope* of f. This leads Aliev and Smyth [1] to try to prove a stronger version of Ruppert's conjecture.

Conjecture 1.1.2 (Aliev-Smyth). *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial. Then the number of isolated torsion points on $Z(f) \subset \mathbb{G}_{\mathrm{m}}^n$ can be bounded above by $c_n \operatorname{vol}_n(\Delta)$, where c_n is a constant depending only on n and Δ is the Newton polytope of f.*

For a general polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, these conjectures imply that the number of isolated torsion points on $Z(f)$ is bounded above by

$$c_n \ \deg(f)^n. \tag{1.1.2}$$

Moreover, this bound implies that the degree of the j-equidimensional part of $\overline{Z(f)}_{\mathrm{tors}}$ is bounded above by $c_{n,j}\delta^{n-j}$, where $c_{n,j}$ is a constant depending only on n and j, see [73, Corollary 11].

In fact, Aliev and Smyth [1] extend Beukers and Smyth's algorithm to higher dimensions and obtained a bound, which however remains far from the conjectured one. For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, they bound the number of maximal torsion cosets in V by

$$\kappa_1(n) \deg(f)^{\kappa_2(n,\delta)} \tag{1.1.3}$$

where

$$\kappa_1(n) = n^{\frac{3}{2}(2+n)5^n} \quad \text{and} \quad \kappa_2(n, \delta) = \frac{1}{16}(49 \cdot \delta^{n-2} - 4n - 9).$$

For sparse representation of polynomials, Leroux [54] gives an algorithm to compute the maximal torsion cosets in $V \subset \mathbb{G}_{\mathrm{m}}^n$. As a consequence, if V can be defined by k polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ with at most r nonzero coefficients, then the number of maximal torsion cosets in V can be bounded above by

$$(r!)^k \exp\Big(3(n+1)\sqrt{kr\log(kr)}\Big).$$

Restricting to the case of dense polynomials, this bound is comparable to that of (1.1.3).

Much sharper bounds follow, as a particular case, from the study of the logarithmic Weil height of points in $\mathbb{G}_{\mathrm{m}}^n$. In fact, the points of zero Weil height are the torsion points, hence bounds on the number of (isolated) points of sufficiently small height yield automatically bounds on the number of (isolated) torsion points. By these means, for a subvariety $V \subset \mathbb{G}_{\mathrm{m}}^n$ defined by polynomials of degree at most δ, David and Philippon [28] and Rémond [70], among others, obtain polynomial upper bounds in δ on the number of maximal torsion cosets in V. Most notably, Amoroso and Viada's results on the essential minimum of V bear the following bounds [3, Corollary 5.4]:

$$\deg(V_{\mathrm{tors}}^j) \leq \left(\delta(200n^5\log(n^2\delta))^{(n-k)n(n-1)}\right)^{n-j},$$

where V_{tors}^{j} is the union all the irreducible components of $\overline{V_{\text{tors}}}$ that are of dimension j, and k is the codimension of V. In particular, if V is a hypersurface in $\mathbb{G}_{\mathrm{m}}^{n}$, the value δ can be taken as the degree of V, and the number of isolated torsion points in V is bounded above by

$$\#(V_{\text{tors}}^{0}) \leq \delta^{n}\ (200n^{5}\log(n^{2}\delta))^{n^{2}(n-1)^{2}}.$$

This bound gives (1.1.2) up to a logarithmic factor.

In this chapter we detail a geometric version of the approach of Beukers and Smyth (Lemma 1.2.5 and Proposition 1.2.6). By algebraic interpolation, using upper and lower bounds on the Hilbert function by Chardin [23], and Chardin and Philippon [24], we obtain hypersurfaces containing the torsion of the variety (Theorem 1.2.16). The first main result (Theorem 1.2.18) follows from adapting the induction techniques introduced by Amoroso and Viada. Given a d-dimensional variety $V \subset \mathbb{G}_{\mathrm{m}}^{n}$ defined by polynomials of degree at most δ, this theorem states that

$$\deg(V_{\text{tors}}^{j}) \leq c_{n}\delta^{n-j} \tag{1.1.4}$$

for every $j = 0, \ldots, d$, where $c_{n} = ((2n-1)(n-1)(2^{2n}+2^{n+1}-2))^{nd}$. Applied to a general hypersurface of degree δ, this proves the bound in (1.1.2).

There is a direct approach to deduce Ruppert's conjecture from (1.1.4), via algebraic group homomorphisms (Corollary 1.2.19). However this method cannot be applied to prove Aliev-Smyth's conjecture. The keystone to obtain this second conjecture from (1.1.4) is a result of John [44] which gives a mean of comparing the volume of a convex polytope with the one of the ellipsoid of smallest volume containing it (John's ellipsoid). Then, by introducing a notion of degree related to a convex polytope (Definition 1.2.22), we get the second main result (Theorem 1.2.23). In particular, given a full-dimensional convex polytope $\Delta \subset \mathbb{R}^{n}$, and a variety $V \subset \mathbb{G}_{\mathrm{m}}^{n}$ defined by polynomials with Newton polytope contained in Δ, this theorem implies that

$$\deg(V_{\text{tors}}^{0}) \leq c_{n}2^{n}n^{2n}\omega_{n}^{-1}\ \mathrm{vol}_{n}(\Delta), \tag{1.1.5}$$

where c_{n} is the constant in (1.1.4), and ω_{n} is the volume of the n-sphere.

Given $f \in \overline{\mathbb{Q}}[x_{1}, \ldots, x_{n}]$ a polynomial of multidegree $(d_{1}, \ldots, d_{n})$, $d_{i} > 0$, we can take Δ to be the n-orthotope $[0, d_{1}] \times \cdots \times [0, d_{n}]$, and then (1.1.5) gives Ruppert's conjecture (Conjecture 1.1.1). Moreover, it suffices to take Δ as the Newton polytope of f to prove Aliev-Smyth's conjecture (Conjecture 1.1.2).

1.1.2 The case of Abelian varieties

The Manin-Mumford conjecture is most notably known for its abelian formulation. Let A be an abelian variety of dimension g defined over a number field. A *torsion point* is an element of finite order with respect to the additive group law of A. For $k \in \mathbb{N}$, we denote by $A[k]$ the group of torsion points of order dividing k, which is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^{2g}$. We write

$$A_{\mathrm{tors}} = \bigcup_{k\in\mathbb{N}} A[k]$$

for the *torsion group* of A.

The abelian statement of Manin-Mumford's conjectre asserts that for a given subvariety V of A, the Zariski closure of $V \cap A_{\mathrm{tors}}$ is a finite union, where each member is a translate of abelian subvarieties of A by a point of finite order. A first partial result is given by Bogomolov [7] for the p^∞-torsion, that is $\cup_{n\geq 1} A[p^n]$. Later, Raynaud proves the conjecture in [68] for the case of a curve embedded in its Jacobian, and in [69] for the general dimension case. Moreover, Hindry [42] gives a general result in which A can be replaced by any algebraic commutative group, in particular a semiabelian variety.

For the case of a smooth, irreducible, projective curve C of genus $g \geq 2$ embedded in its Jacobian $J(C)$, there are many different effective bounds on the number of torsion points in C, for instance Raynaud [68], Coleman [25], and Hindry [41]. Using p-jets, and under some ramification conditions on a prime $p \geq 2g+1$, Buium [15] obtains that

$$\#C_{\mathrm{tors}} \leq g! p^{4g} 3^g (p(2g-2)+6g),$$

responding to a question posed by Mazur [61, p.234] on a uniform bound depending only on the genus of the curve, and on the prime p.

For the sequel, let us fix a closed immersion $\iota : A \hookrightarrow \mathbb{P}^n$ into a projective space of some dimension n. Given a subvariety $V \subset A$, we focus on effective bounds on the number of maximal torsion cosets in V, which correspond to abelian subvarieties of A translated by torsion points of A that are maximal with respect to the inclusion. Hindry's approach in [42] yields already an effective bound (up to a constant related to Galois representations), which is not made explicit. Later, Bombieri and Zannier [10] show that the number of maximal torsion cosets in V can be bounded just in terms of the degree of V by ι, and data coming from A.

By means of new model-theoretic methods, Hrushovski [43] bounds the number of maximal torsion cosets in V by

$$c \ \deg(V)^e, \tag{1.1.6}$$

where c and e depend only on A (in fact they are doubly exponential in parameters coming from A), and $\deg(V)$ denotes the degree of the Zariski closure of the image of V by the fixed immersion ι.

Given the results in the toric case regarding the dependence on the degree (Theorem 1.2.18), it is a natural question to ask if one can improve the exponent e in (1.1.6), with the cost of incrementing the multiplicative coefficient c. Given V a subvariety of A defined in $\mathbb{P}^n$ as the intersection of finite number of hypersurfaces of degree at most δ, one expects to bound the number of maximal torsion cosets in V by

$$c\,\delta^{g}, \tag{1.1.7}$$

where g is the dimension of A, and c is a constant only depending on A. To prove such a statement, our aim is to adapt the techniques of the toric case.

Let K be a the number field such that A is defined over K. Using the results of Bogomolov [7] and Serre [80] on the homotheties in the image of the absolute Galois group of K by the l-adic representations, one has that there exists an integer $\mathrm{c}(A)$ which depends only on A (and K) such that for every point $P \in A$ of finite order, and any integer k prime to the order of P, there is a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that

$$P^{\sigma} = [k^{\mathrm{c}(A)}]\,P. \tag{1.1.8}$$

This classical approach to the Manin-Mumford conjecture was first proposed by Lang [48], and has since proven to be a succesful tool (see [69] and [42]).

In this chapter we retake this approach to the proof Manin-Mumford's conjecture in the abelian setting. The main idea is to set analogies with the toric version, and extend them to Abelian varieties. The much more complicated structure of torsion points and their of definition is however the main problem in establishing such analogies.

Let V be a subvariety of A. By means of a careful choice of homotheties coming from Galois automorphisms, we are able to give an explicit construction of an auxiliary variety containing the torsion of V (Propositions 1.3.7 and 1.3.8). The first result of interest arises when considering V to be a curve of genus $g \geq 2$. In this case, for an irreducible algebraic curve $C \subset A$, we obtain the following Abelian analogue to Beukers and Smyth's result:

$$\#\, C_{\mathrm{tors}} \leq (2^{2g\mathrm{c}(A)+4g-2\mathrm{c}(A)}\,\mathrm{c}(A)^{2g} + 2^{2g+1} - 1)\deg(C)^2;$$

see Proposition 1.3.9 and the remark that follows.

To further extend our result to higher dimensional varieties, we proceed by mimicking the process followed in the toric case. By identifying V, and A with their images in $\mathbb{P}^n$, we use relative versions of upper and lower bounds on the Hilbert function (due again to Chardin [23], and Chardin and Philippon [24]), to obtain an interpolating hypersurface in $\mathbb{P}^n$ that intersects V (Proposition 1.3.13). In addition, our result bounds the degree of this hypersurface in terms of degree δ of the hypersurfaces in $\mathbb{P}^n$ such that V is defined as the intersection of them.

This leads to the third main result of this chapter (Theorem 1.3.14) which states the following. If $\dim(A) = g$, and V is a d-dimensional subvariety of A that can be defined in $\mathbb{P}^n$ as the intersection of hypersurfaces of degree at most δ, then

$$\deg(V^{j}_{\mathrm{tors}}) \leq c_j\,\delta^{\,g-j},$$

for every $j = 0, \dots, d$, where $c_j = ((2g-1)(g-1)(2^{2g(2+\mathrm{c}(A))+2}\,\mathrm{c}(A)^{2g} + 2^{2g+2} - 2))^{(g-j)d}$. Moreover, this explicit version of Manin-Mumford's conjecture is effective, up to the non-effective constant $c(A)$.

1.2 Bounds for the toric Manin-Mumford

For the length of this chapter $\mathbb{G}_{\mathrm{m}}^n$ denotes $(\mathbb{C}^\times)^n$. If not specified, we consider $\mathbb{G}_{\mathrm{m}}^n$ naturally embedded into $\mathbb{P}^n$ by $(x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$. When considering subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ they are defined over $\mathbb{C}$ unless stated otherwise. Moreover, when we say that a variety is irreducible, we imply it to be irreducible over $\mathbb{C}$.

1.2.1 Geometric extrapolation of the torsion points

Let $\boldsymbol{x} \in \mathbb{G}_{\mathrm{m}}^n$ be a point and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ be an integer vector, we adopt the multi-index notation

$$\boldsymbol{x}^{\boldsymbol{\lambda}} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

So, a family of vectors vectors $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r \in \mathbb{Z}^n$, $r > 0$, induces an (algebraic group) homomorphism

$$\mathbb{G}_{\mathrm{m}}^n \longrightarrow \mathbb{G}_{\mathrm{m}}^r, \quad \boldsymbol{x} \longmapsto (\boldsymbol{x}^{\boldsymbol{\lambda}_1}, \dots, \boldsymbol{x}^{\boldsymbol{\lambda}_r}). \tag{1.2.1}$$

In fact, this defines is a bijection between integer matrices $\mathcal{M}_{r,n}(\mathbb{Z})$ and (algebraic group) homomorphisms $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}}^n, \mathbb{G}_{\mathrm{m}}^r)$ by taking the $\boldsymbol{\lambda}_j$'s in (1.2.1) as the row vectors of the matrix in $\mathcal{M}_{r,n}(\mathbb{Z})$. In particular, for any $l \in \mathbb{Z}$, we define the *multiplication map* by l as the endomorphism

$$\begin{aligned} [l] : \mathbb{G}_{\mathrm{m}}^n &\longrightarrow \mathbb{G}_{\mathrm{m}}^n \\ (x_1, \dots, x_n) &\longmapsto (x_1^l, \dots, x_n^l) \end{aligned}$$

which corresponds to the diagonal matrix $l \cdot \mathrm{Id} \in \mathcal{M}_{n \times n}(\mathbb{Z})$. Hence, we may express the subgroup of the k-torsion points of $\mathbb{G}_{\mathrm{m}}^n$ as

$$\mu_k^n = \{\boldsymbol{x} \in \mathbb{G}_{\mathrm{m}}^n \mid [k]\boldsymbol{x} = \mathbf{1}\}.$$

Let Λ be a subgroup of $\mathbb{Z}^n$. We denote by $\Lambda^{\mathrm{sat}} = (\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n$ the *saturation* of Λ, and we call $[\Lambda^{\mathrm{sat}} : \Lambda]$ the *index* of Λ. In particular, we say that Λ is *saturated* if $[\Lambda^{\mathrm{sat}} : \Lambda] = 1$. We define the algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$ associated to Λ as

$$H_\Lambda = \{\boldsymbol{x} \in \mathbb{G}_{\mathrm{m}}^n \mid \boldsymbol{x}^{\boldsymbol{\lambda}} = 1, \forall \boldsymbol{\lambda} \in \Lambda\}.$$

The following result sums up the relation between subgroups of $\mathbb{Z}^n$ and algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$.

Theorem 1.2.1. *The map $\Lambda \mapsto H_\Lambda$ is a dimension reversing bijection between subgroups of $\mathbb{Z}^n$ and algebraic subgroups of $\mathbb{G}_\mathrm{m}^n$. A subgroup H_Λ is irreducible if and only if Λ is saturated. Moreover, for any two subgroups Λ and Λ' we have $H_\Lambda \cdot H_{\Lambda'} = H_{\Lambda\cap\Lambda'}$, and $H_\Lambda \cap H_{\Lambda'} = H_{\Lambda+\Lambda'}$.*

Proof. See [8, Proposition 3.2.7 and Theorem 3.2.19]. $\square$

A homomorphism $\mathbb{G}_\mathrm{m}^n \to \mathbb{G}_\mathrm{m}^r$ defines an algebraic subgroup of $\mathbb{G}_\mathrm{m}^n$ by means of the kernel. It is also possible to build a homomorphism with a fixed kernel.

Corollary 1.2.2. *Let H be an algebraic subgroup of $\mathbb{G}_\mathrm{m}^n$ of dimension $n-r$. We can write $H = F \cdot H^0$, where F is a finite subgroup of $\mathbb{G}_\mathrm{m}^n$, and H^0 denotes the connected component of H containing $\mathbf{1}$. Moreover, there exists an isogeny*

$$\overline{\varphi} : \mathbb{G}_\mathrm{m}^n \longrightarrow H^0 \times \mathbb{G}_\mathrm{m}^r,$$

such that $\mathrm{Ker}(\overline{\varphi}) = F$*, and $\varphi|_{H^0}(H) : H^0 \to H^0 \times \{\mathbf{1}\}$ is the identity.*

Proof. By Theorem 1.2.1, there exists a lattice Λ such that $H = H_\Lambda$. Write $\Lambda = \Lambda^{\mathrm{sat}} \cap \Lambda^*$, where Λ^* is a lattice of full dimension. Then $H = F \cdot H^0$, with $H^0 = H_{\Lambda^{\mathrm{sat}}}$ and $F = H_{\Lambda^{\mathrm{sat}}}$. Since Λ^* is full dimensional, F is a finite subgroup of $\mathbb{G}_\mathrm{m}^n$. Moreover, there is an isogeny $\mathbb{G}_\mathrm{m}^n \to \mathbb{G}_\mathrm{m}^n$, such that its kernel is F. This allows us to reduce to the case when $H = H^0$.

By Theorem 1.2.1, there exists a unique saturated lattice $\Lambda \subset \mathbb{Z}^n$ such that $H = H_\Lambda$. Take a complementary subgroup $\Lambda' \subset \mathbb{Z}^n$, that is a saturated lattice such that $\Lambda \cap \Lambda' = \{\mathbf{0}\}$ and $\Lambda + \Lambda' = \mathbb{Z}^n$. Then $H_{\Lambda'}$ is irreducible, and so $H_{\Lambda'} \cong \mathbb{G}_\mathrm{m}^r$. Also by Theorem 1.2.1, we have that $\mathbb{G}_\mathrm{m}^n = H_{\{\mathbf{0}\}} = H_{\Lambda\cap\Lambda'}$, and $\mathbf{1} = H_{\mathbb{Z}^n} = H_{\Lambda+\Lambda'}$. Then we have an isomorphism $\mathbb{G}_\mathrm{m}^n = H \times H_{\Lambda'} \cong H \times \mathbb{G}_\mathrm{m}^r$. $\square$

A subgroup of $\mathbb{G}_\mathrm{m}^n$ with special interest for this chapter is the stabilizer of a variety. For a subvariety V of $\mathbb{G}_\mathrm{m}^n$, we define the *stabilizer* of V as

$$\mathrm{Stab}(V) = \{\boldsymbol{\xi} \in \mathbb{G}_\mathrm{m}^n \mid \boldsymbol{\xi} V = V\}.$$

First, notice that $\dim(\mathrm{Stab}(V)) \leq \dim(V)$. Moreover, the dimensions coincide if and only if V is a translate of an algebraic subgroup of $\mathbb{G}_\mathrm{m}^n$. In this latter case either $\overline{V_{\mathrm{tors}}} = V$, or there are no torsion points in V.

The following fact should also be highlighted. If $\psi : \mathbb{G}_\mathrm{m}^n \to \mathbb{G}_\mathrm{m}^r$ is a surjective homomorphism, and $W \subset \mathbb{G}_\mathrm{m}^r$ a variety, then $\psi^{-1}(\mathrm{Stab}(W)) = \mathrm{Stab}(\psi^{-1}(W))$.

By means of the homomorphism appearing in Corollary 1.2.2, we associate to V a subvariety of some $\mathbb{G}_\mathrm{m}^r$ which has trivial stabilizer. The following result is a direct consequence of this corollary and illustrates some useful properties of this homomorphism.

Corollary 1.2.3. *Let V be a subvariety of $\mathbb{G}_\mathrm{m}^n$, and $r = \mathrm{codim}(\mathrm{Stab}(V))$. There exists a homomorphism $\varphi : \mathbb{G}_\mathrm{m}^n \to \mathbb{G}_\mathrm{m}^r$ such that* $\mathrm{Ker}(\varphi) = \mathrm{Stab}(V)$*, satisfying the following properties:*

(i) $\varphi(V)$ *is a subvariety of* $\mathbb{G}_{\mathrm{m}}^r$ *with trivial stabilizer;*

(ii) $\varphi^{-1}(\varphi(V)) = V$*;*

(iii) $\varphi^{-1}(\boldsymbol{\eta})V = \boldsymbol{\eta}_0 V$*, for every* $\boldsymbol{\eta} \in \mathbb{G}_{\mathrm{m}}^r$ *and for any* $\boldsymbol{\eta}_0 \in \varphi^{-1}(\boldsymbol{\eta})$*.*

Proof. Write $\mathrm{Stab}(V) = F \cdot \mathrm{Stab}(V)^0$, with F a finite subgroup of $\mathbb{G}_{\mathrm{m}}^n$ and $\mathrm{Stab}(V)^0$ the irreducible component passing through $\mathbf{1}$. By Corollary 1.2.2, one has an isogeny $\overline{\varphi} : \mathbb{G}_{\mathrm{m}}^n \to \mathrm{Stab}(V)^0 \times \mathbb{G}_{\mathrm{m}}^r$, such that $\mathrm{Ker}(\overline{\varphi}) = F$. Since $\overline{\varphi}$ is an isogeny, the image of V by $\overline{\varphi}$ is a variety. Moreover, if $\boldsymbol{\xi}\overline{\varphi}(V) = \overline{\varphi}(V)$, for a $\boldsymbol{\xi} \in \mathrm{Stab}(V)^0 \times \mathbb{G}_{\mathrm{m}}^r$, by taking preimages $\boldsymbol{\xi}' \cdot F \cdot V = F \cdot V$, for some $\boldsymbol{\xi}' \in \overline{\varphi}^{-1}(\boldsymbol{\xi})$. In particular, since $F \subset \mathrm{Stab}(V)$, we have $\boldsymbol{\xi} \in \overline{\varphi}(\mathrm{Stab}(V))$, and therefore $\mathrm{Stab}(\overline{\varphi}(V)) = \overline{\varphi}(\mathrm{Stab}(V)) = \mathrm{Stab}(V)^0 \times \{\mathbf{1}\}$. Hence, $\overline{\varphi}(V)$ is of the form $\mathrm{Stab}(V)^0 \times V'$, where V' is a subvariety of $\mathbb{G}_{\mathrm{m}}^r$. Then, the homomorphism φ is obtained from $\overline{\varphi}$ by taking the projection to $\mathbb{G}_{\mathrm{m}}^r$. The properties in the statement follow then by construction. □

There is a remarkable relation between the stabilizer and torsion cosets in V. To illustrate this, let ωH be a torsion coset in V (not necessarily maximal) and let $\mathrm{Stab}(V)^0$ be the connected component of $\mathrm{Stab}(V)$ containing $\mathbf{1}$. Then $\bigcup_{\boldsymbol{\xi} \in \mathrm{Stab}(V)^0} \boldsymbol{\xi} \cdot (\boldsymbol{\omega} H)$ is a torsion coset in V that contains ωH. In particular, every maximal torsion coset in V has dimension at least $\dim(\mathrm{Stab}(V))$, and its subtorus contains $\mathrm{Stab}(V)^0$.

To fix notations, given a variety $V \subset \mathbb{G}_{\mathrm{m}}^n$ and an automorphism $\phi \in \mathrm{Aut}(\overline{\mathbb{C}}/\mathbb{Q})$, we denote by V^{ϕ} the variety obtained by applying ϕ to the coefficients the polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ defining V.

Torsion points, being essentially vectors of roots of unity, are defined over cyclotomic extensions of $\mathbb{Q}$. Hence, any Galois automorphism fixing the maximal abelian extension of $\mathbb{Q}$ leaves invariant the torsion cosets of $\mathbb{G}_{\mathrm{m}}^n$. This observation gives the following result:

Proposition 1.2.4. *Let* $V \subset \mathbb{G}_{\mathrm{m}}^n$ *be an irreducible variety of positive dimension defined over a finite Galois extension* $\mathbb{K}$ *of* $\mathbb{Q}$*, that is not contained in* $\mathbb{Q}^{ab}$*. There exists a non trivial Galois automorphism* $\varsigma \in \mathrm{Gal}(\mathbb{K}/(\mathbb{K} \cap \mathbb{Q}^{ab}))$*, such that*

$$\overline{V_{tors}} \subset V \cap V^{\varsigma} \subsetneq V.$$

It is so important to singularize the study of varieties defined over $\mathbb{Q}^{\mathrm{ab}}$. By the Kronecker-Weber theorem, whenever we have an abelian extension $\mathbb{K}$ of $\mathbb{Q}$, we have that $\mathbb{K}$ is contained in a cyclotomic extension of $\mathbb{Q}$. In fact, there is a unique minimal natural number, which we denote by $N_{\mathbb{K}}$, such that the $N_{\mathbb{K}}$-th cyclotomic field is the minimal cyclotomic extension of $\mathbb{Q}$ containing $\mathbb{K}$, see for instance [62, Theorem 4.27(v)]. Given V a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ defined over an abelian extension of $\mathbb{Q}$, we choose the minimal natural number N as

$$N = \min_{\boldsymbol{\xi} \in \mu_{\infty}^n} \{N_{\mathbb{K}} \mid \ \mathbb{K} \text{ is the field of definition of } \boldsymbol{\xi} \cdot V\}. \tag{1.2.2}$$

In particular, notice that if $N \equiv 2 \pmod 4$, then $\mathbb{Q}(\zeta_N) = \mathbb{Q}(\zeta_{N/2})$. Therefore, we can always choose $N \not\equiv 2 \pmod 4$. We adopt the notation ζ_N for a primitive N-th root of unity, and $\mathbb{Q}(\zeta_N)$ for the N-th cyclotomic extension of $\mathbb{Q}$.

Remark. *Notice that the value of $N_{\mathbb{K}}$ (and henceforth also the value of N) is the same for V and $\varphi(V)$, with φ as in Corollary 1.2.3. This follows from the fact that two varieties $V, W \subset \mathbb{G}_{\mathrm{m}}^n$ with the same stabilizer define the same homomorphism φ, and then $V = W$ if and only if $\varphi(V) = \varphi(W)$. Fixed an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, take $W = V^\sigma$. Since the stabilizer is an algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$, it is defined over $\mathbb{Q}$ by Theorem 1.2.1, hence $\mathrm{Stab}(V^\sigma) = \mathrm{Stab}(V)$. Therefore $V = V^\sigma$ if and only if $\varphi(V) = \varphi(V^\sigma) = \varphi(V)^\sigma$. From here we deduce that V and $\varphi(V)$ are defined over the same cyclotomic extensions of $\mathbb{Q}$.*

For the remaining of this section, N, N', M, M', l and l' represent positive integers.

Lemma 1.2.5. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety whose field of definition is an abelian extension $\mathbb{K}$ of $\mathbb{Q}$. Let $\boldsymbol{\omega} \in V$ be a torsion point.*

1. *If $4 \nmid N_{\mathbb{K}}$, one of the following is true:*
   (a) *there exists a 2-torsion point $\boldsymbol{\eta} \in \mu_2^n \setminus \{\mathbf{1}\}$ such that $\boldsymbol{\eta} \cdot \boldsymbol{\omega} \in V$;*
   (b) *there exists a 2-torsion point $\boldsymbol{\eta} \in \mu_2^n$ such that $\boldsymbol{\eta} \cdot [2]\boldsymbol{\omega} \in V^\sigma$, where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{N_{\mathbb{K}}})/\mathbb{Q})$ is the Galois automorphism mapping $\zeta_{N_{\mathbb{K}}} \mapsto \zeta_{N_{\mathbb{K}}}^2$.*
2. *If $N_{\mathbb{K}} = 4N'$, one of the following is true:*
   (c) *there exists a 2-torsion point $\boldsymbol{\eta} \in \mu_2^n \setminus \{\mathbf{1}\}$ such that $\boldsymbol{\eta} \cdot \boldsymbol{\omega} \in V$;*
   (d) *there exists a 2-torsion point $\boldsymbol{\eta} \in \mu_2^n$ such that $\boldsymbol{\eta} \cdot \boldsymbol{\omega} \in V^\tau$ where $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_{N_{\mathbb{K}}})/\mathbb{Q})$ is a Galois automorphism mapping $\zeta_{N_{\mathbb{K}}} \mapsto \zeta_{N_{\mathbb{K}}}^{1+2N'}$.*

Proof. To simplify the presentation, throughout this proof we denote $N_{\mathbb{K}}$ by N. Let l be the order of $\boldsymbol{\omega}$, in particular $\boldsymbol{\omega} \in \mathbb{Q}(\zeta_l)$, and $M = \mathrm{lcm}(N, l)$. We prove separately point 1 and 2.

1. By hypothesis, N is odd. We distinguish 3 cases regarding the parity of l, where the first corresponds to (a) and the other two to (b).
   (i) If $l = 4l'$, then $M = 4M'$. In particular, we have $\gcd(1 + 2M', M) = 1$. Therefore, we can take a Galois automorphism $\widetilde{\tau} \in \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ mapping $\zeta_M \mapsto \zeta_M^{1+2M'}$. Since $2M' \equiv 2l' \pmod l$, we have that $\widetilde{\tau}$ maps $\zeta_l \mapsto \zeta_l^{1+2l'}$. On the other hand, N is odd so $N|M'$ and ζ_N is invariant under the action of $\widetilde{\tau}$. Hence $V^{\widetilde{\tau}} = V$ and $[1 + 2l']\boldsymbol{\omega} \in V$. Choosing $\boldsymbol{\eta} = [2l']\boldsymbol{\omega} \in \mu_2^n \setminus \{\mathbf{1}\}$, (a) holds.

(ii) If $l = 2l'$ with $2 \nmid l'$, then $M = 2M'$ with $2 \nmid M'$. In particular, we have $\gcd(2+M', M) = 1$. Therefore, we can extend σ to a Galois automorphism in $\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$, mapping $\zeta_M \mapsto \zeta_M^{2+M'}$ (this extends σ because $N \mid M'$, since N is odd). Since $M' \equiv l' \pmod{l}$, we have that σ maps $\zeta_l \mapsto \zeta_l^{2+l'}$. Hence $[2+l']\boldsymbol{\omega} \in V^\sigma$. Choosing $\boldsymbol{\eta} = [l']\boldsymbol{\omega} \in \mu_2^n$, (b) holds.

(iii) If $2 \nmid l$, then $2 \nmid M$. We have that σ can be extended to a Galois automorphism in $\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ mapping $\zeta_M \mapsto \zeta_M^2$. In particular, σ maps $\zeta_l \mapsto \zeta_l^2$. Hence $[2]\boldsymbol{\omega} \in V^\sigma$. Choosing $\boldsymbol{\eta} = \mathbf{1}$, (b) holds.

2. By hypothesis $4 \mid N$, so we also have $4 \mid M$. Write $N = 4N'$ and $M = 4M'$. Let $\widetilde{\tau}$ be an automorphism in $\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ mapping $\zeta_M \mapsto \zeta_M^{1+2M'}$. Let v_2 denote the 2-adic valuation. We distinguish 2 cases by comparing the 2-adic valuations of N and l, corresponding to (c) and (d) respectively.

(i) If $v_2(N) < v_2(l)$, then $N \mid 2M'$ and $l \nmid 2M'$. Write $l = 4l'$. Since $2M' \equiv 2l' \pmod{l}$, we have that $\widetilde{\tau}$ maps $\zeta_l \mapsto \zeta_l^{1+2l'}$. On the other hand, $2M' \equiv 0 \pmod{N}$ and so $\widetilde{\tau}$ fixes $\mathbb{Q}(\zeta_N)$. Hence $V^{\widetilde{\tau}} = V$ and $[1+2l']\boldsymbol{\omega} \in V$. Choosing $\boldsymbol{\eta} = [2l']\boldsymbol{\omega} \in \mu_2^n \setminus \{\mathbf{1}\}$, we obtain that (c) holds.

(ii) If $v_2(N) \geq v_2(l)$, then $N \nmid 2M'$. We have that either $2M' \equiv 0 \pmod{l}$ or, if not, $2M' \equiv l/2 \pmod{l}$, therefore $[2M']\boldsymbol{\omega} \in \mu_2^n$. On the other hand, $2N' \equiv 2M' \pmod{N}$ and so $\widetilde{\tau}$ is an extension of τ. Hence $[2M'+1]\boldsymbol{\omega} \in V^\tau$. Choosing $\boldsymbol{\eta} = [2M']\boldsymbol{\omega}$, (d) holds. $\square$

Remark. *The particular case when the field of definition of V is $\mathbb{Q}$ is covered in Lemma 1.2.5. It corresponds to point 1, taking $N = 1$ and $\sigma = \mathrm{Id}$.*

When considering an irreducible variety $V \subset \mathbb{G}_\mathrm{m}^n$ defined over an abelian extension of $\mathbb{Q}$, this lemma lays the groundwork for an equivalent result to Proposition 1.2.4. We provide an explicit construction of a variety V' containing $\overline{V_{\mathrm{tors}}}$ but not V. For this last condition a good control over the stabilizer of V is necessary.

Proposition 1.2.6. *Let $V \subset \mathbb{G}_\mathrm{m}^n$ be an irreducible variety of positive dimension, defined over an abelian extension $\mathbb{K}$ of $\mathbb{Q}$ such that $\overline{V_{tors}} \neq V$. Let N be as in* (1.2.2) *and suppose that $N = N_\mathbb{K}$. Let $r = \mathrm{codim}(\mathrm{Stab}(V))$ and $\varphi : \mathbb{G}_\mathrm{m}^n \to \mathbb{G}_\mathrm{m}^r$ a homomorphism such that* $\mathrm{Stab}(V) = \mathrm{Ker}(\varphi)$*.*

1. If $4 \nmid N$, then

$$\overline{V_{tors}} \subset V' = \bigcup_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} (\varphi^{-1}(\boldsymbol{\eta})V) \cup \bigcup_{\boldsymbol{\eta} \in \mu_2^r} [2]^{-1}(\varphi^{-1}(\boldsymbol{\eta})V^\sigma),$$

where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, mapping $\zeta_N \mapsto \zeta_N^2$. Moreover $V' \cap V \subsetneq V$.

2. If $N = 4N'$, then

$$\overline{V_{tors}} \subset V' = \bigcup_{\boldsymbol{\eta}\in\mu_2^r\setminus\{\mathbf{1}\}} (\varphi^{-1}(\boldsymbol{\eta})V) \cup \bigcup_{\boldsymbol{\eta}\in\mu_2^r} (\varphi^{-1}(\boldsymbol{\eta})V^{\tau}),$$

where $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, mapping $\zeta_N \mapsto \zeta_N^{1+2N'}$. Moreover $V' \cap V \subsetneq V$.

The expressions of V' in the proposition are set-theoretical, and in fact they are the finite union of $(2^r - 1) + 2^r$ varieties, see point (iii) of Corollary 1.2.3.

Proof. To show the inclusion $\overline{V_{\mathrm{tors}}} \subset V'$, it is enough to prove that every torsion point in V lies also in V'. Torsion is preserved by homomorphisms, so $\varphi(V_{\mathrm{tors}}) = \varphi(V)_{\mathrm{tors}}$, and we may apply Lemma 1.2.5 to $\varphi(V)$. If $4 \nmid N$, then by taking the union of all the varieties that come from points (a) and (b) of said lemma, we obtain that

$$\varphi(\overline{V_{\mathrm{tors}}}) \subset \bigcup_{\boldsymbol{\eta}\in\mu_2^r\setminus\{\mathbf{1}\}} \boldsymbol{\eta}\cdot\varphi(V) \cup \bigcup_{\boldsymbol{\eta}\in\mu_2^r} [2]^{-1}(\boldsymbol{\eta}\cdot\varphi(V^{\sigma})).$$

By taking the preimage of the variety on the right by φ we obtain the V' in the statement. In a similar way, we may apply case 2 in Lemma 1.2.5 to prove the second inclusion of the proposition.

To prove that $V \cap V' \subsetneq V$, we can assume that V has a trivial stabilizer since φ does not change the value of N. So $\varphi = \mathrm{Id}$. We proceed by showing that V is not contained in any of the varieties that come from the lemma. First, since V has trivial stabilizer by hypothesis, $\boldsymbol{\eta}\cdot V \neq V$ for every $\boldsymbol{\eta} \in \mu_2^n \setminus \{\mathbf{1}\}$, and so $V \cap \boldsymbol{\eta}\cdot V \subsetneq V$ for all such $\boldsymbol{\eta}$'s. This deals with the varieties coming from (a) and (c).

To see that $V \subsetneq [2]^{-1}(\boldsymbol{\eta}\cdot V^{\sigma})$ for all $\boldsymbol{\eta} \in \mu_2^n$, we do it by explicitly computing the degrees. Assume that $V \subset [2]^{-1}(\boldsymbol{\eta}\cdot V^{\sigma})$, then $[2](\boldsymbol{\eta}_0 \cdot V) \subset \boldsymbol{\eta}\cdot V^{\sigma}$ for every $\boldsymbol{\eta}_0 \in \mathrm{Ker}[2] = \mu_2^n$. Thus

$$\bigcup_{\boldsymbol{\eta}_0\in\mu_2^n} \boldsymbol{\eta}_0 \cdot V \subset [2]^{-1}(\boldsymbol{\eta}\cdot V^{\sigma}).$$

Since V has trivial stabilizer, the variety on the left is a union of 2^n distinct varieties and so it has degree $2^n \deg(V)$. On the other side the variety has degree $2^{\mathrm{codim}(V)}\deg(V)$, see [42, Lemme 6(i)]. The contradiction arises from the fact that $\mathrm{codim}(V) < n$. This deals the varieties arising from (b).

It is left to proof that $V \neq \boldsymbol{\eta}\cdot V^{\tau}$ for all $\boldsymbol{\eta} \in \mu_2^n$, which correspond to the varieties coming from (d). To do so, assume for instance that there is an equality and choose some $\boldsymbol{\xi} \in \mu_N^n \setminus \{\mathbf{1}\}$ such that $[2N']\boldsymbol{\xi} = \boldsymbol{\eta}$. Then $\boldsymbol{\xi}^{\boldsymbol{\tau}} = \boldsymbol{\xi}\cdot\boldsymbol{\eta}$ and $(\boldsymbol{\xi}\cdot V)^{\tau} = \boldsymbol{\xi}^{\boldsymbol{\tau}}\cdot V^{\tau} = \boldsymbol{\xi}\cdot V$. This would mean that $\boldsymbol{\xi}\cdot V$ is stable by τ and so it is defined over $\mathbb{Q}(\zeta_N)^{\tau} = \mathbb{Q}(\zeta_{N/2})$, which contradicts the minimality of N (1.2.2), and finishes the proof. □

1.2.2 Algebraic interpolation

Let $V \subset \mathbb{G}_\mathrm{m}^n$ be a variety of positive dimension. When V is irreducible, by using Propositions 1.2.4 and 1.2.6 we can explicitly construct an equidimensional variety V' of the same dimension that contains $\overline{V_{\mathrm{tors}}}$ and such that $V \cap V' \subsetneq V$. The degree of V' can be easily computed. In the case that V is not defined over $\mathbb{Q}^{\mathrm{ab}}$, the degree of V' is the same as the one of V. On the other hand, if the field of definition of V is an abelian extension of $\mathbb{Q}$ we may use [42, Lemme 6(i)] and obtain the following two cases depending on the parity of N:

1. if $2 \nmid N$, then $\deg(V') = (2^r - 1)\deg(V) + 2^r\, 2^{\mathrm{codim}(V)} \deg(V)$,
2. if $4 \mid N$, then $\deg(V') = (2^r - 1)\deg(V) + 2^r \deg(V)$,

where $r = \mathrm{codim}_{\mathbb{G}_\mathrm{m}^n}(\mathrm{Stab}(V))$. The idea to apply straightforwardly Bézout's theorem yields a bound on the number of maximal torsion cosets. If V is a non-torsion d-dimensional variety defined over $\mathbb{Q}$, such that $\dim(\mathrm{Stab}(V)) = \dim(V) - 1$, we retrieve the optimal bound this method gives:

$$\deg(\overline{V_{\mathrm{tors}}}) \leq (2^{n-d+1} + 2^{2n-2d+1} - 1)\deg(V)^2. \tag{1.2.3}$$

In the particular case when $n = 2$ and V is a curve we have that the number of torsion points of V is at most $11\deg(V)^2$, which corresponds to the bound given by Beukers and Smyth [5], see (1.1.1). However, the iteration of this method does increase the exponent of $\deg(V)$ exponentially, which motivates the use of the following definition.

Definition 1.2.7. Let V be a subvariety of $\mathbb{G}_\mathrm{m}^n$. We define the *degree of definition* of V as the minimal degree δ such that V is the intersection of hypersurfaces of degree at most δ, and we denote it by $\delta(V)$.

We also define the *degree of incomplete definition* of V as the minimal degree δ_0 such that there exists a variety X that is the intersection of hypersurfaces of degree at most δ_0, such that any irreducible component of V is a component of X. We denote it by $\delta_0(V)$.

Lemma 1.2.8. *If $V \subset \mathbb{G}_\mathrm{m}^n$ is defined over $\mathbb{K}$, then $\delta(V)$ and $\delta_0(V)$ can be realized by hypersurfaces defined over $\mathbb{K}$.*

Proof. Let I be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ whose zero set is V, and let $I \otimes \mathbb{C}$ be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ defined by base change. Since $\mathbb{K}$ is perfect we can apply [12, Chapitre 5, §15.5 Théorème 3(d)] to obtain the equality $\sqrt{I \otimes \mathbb{C}} = \sqrt{I} \otimes \mathbb{C}$. Hence the radical ideal $I(V)$ in $\mathbb{C}[x_1, \ldots, x_n]$ defining V equals $\sqrt{I} \otimes \mathbb{C}$, and is defined over $\mathbb{K}$. For $\delta \geq 0$, denote by $I(V)_{\leq\delta} \subset \mathbb{C}[x_1, \ldots, x_n]$ the subspace of the polynomials in $I(V)$ of degree at most δ. Since $I(V)$ is defined over $\mathbb{K}$, also is $I(V)_{\leq\delta}$.

The definition of $\delta(V)$, is equivalent to the minimal δ such that the zero set of $I(V)_{\leq\delta}$ equals V. On the other hand, the definition of $\delta_0(V)$ is equivalent to the minimal δ

such that the zero set of $I(V)_{\leq\delta}$ equals $V \cup W$, for some subvariety $W \subset \mathbb{G}_m^n$ such that $V \not\subset W$. Then the lemma follows from the fact that $I(V)_{\leq\delta}$ is defined over $\mathbb{K}$. □

Let V an equidimensional variety of dimension d. Given a general linear map $\ell : \mathbb{P}^n \to \mathbb{P}^{d+1}$, the image of V by ℓ is a hypersurface of degree at most $\deg(V)$. We can take the pull-back of this hypersurface by ℓ, which gives a hypersurface of degree at most $\deg(V)$ containing V. Then V is (as a set) intersection of all hypersurfaces obtained in this way. This shows that $\delta(V) \leq \deg(V)$. Moreover,

$$\delta_0(V) \leq \delta(V) \leq \deg(V) \leq \delta_0(V)^{\mathrm{codim}(V)},$$

where the first inequality follows from Definition 1.2.7, and the last one from [66, Corollaire 5]. Notice that, when intersecting V with a hypersurface, the definition of δ gives $\delta(V \cap Z) \leq \max\{\delta(V), \deg(Z)\}$, and the same is true for δ_0. The behaviour of δ_0 is however more subtle with regard of the union of varieties. Let us recall first an easy lemma for the degree of definition.

Lemma 1.2.9. *Let $X_1, \ldots, X_t$ be subvarieties of $\mathbb{G}_m^n$. Then*

$$\delta(\bigcup_{i=1}^{t} X_i) \leq \sum_{i=1}^{t} \delta(X_i).$$

Proof. It is enough to prove it for $t = 2$. Let X_1 be defined by polynomials $f_1, \ldots, f_r$ with $\deg(f_i) \leq \delta(X_1)$, and X_2 be defined by $g_1, \ldots, g_s$ with $\deg(g_i) \leq \delta(X_2)$. Then $X_1 \cup X_2$ is defined by the polynomials $f_i g_j$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. □

In general, this result is not true if we use δ_0 instead of δ. To have a similar lemma for δ_0, we must therefore consider more specific varieties. The following is a variation of [3, Lemma 2.5], which takes into account the action of Galois automorphisms on the computation of δ_0 of a variety.

Lemma 1.2.10. *Let V be an irreducible subvariety of $\mathbb{G}_m^n$. Let $M > 2$ be a positive integer, and ζ_M be a primitive M-th root of unity, such that V is defined over $\mathbb{Q}(\zeta_M)$. Let $T \subset \mu_M^n \times \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ be a finite set with t elements. Then*

$$\delta_0(\bigcup_{(\boldsymbol{g},\phi)\in T} \boldsymbol{g}V^{\phi}) \leq t\delta_0(V).$$

Proof. Throughout this proof, we say that an irreducible variety $W \subset \mathbb{G}_m^n$ is imbedded in a variety $X \subset \mathbb{G}_m^n$ if $W \subset X$ but W is not an irreducible component of X.

Notice that for any two $g_1, g_2 \in \mu_M^n$ and any two $\phi_1, \phi_2 \in \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$, we have that $\boldsymbol{g}_2(\boldsymbol{g}_1 V^{\phi_1})^{\phi_2} = \boldsymbol{g}_2\phi_2^{-1}(\boldsymbol{g}_1)V^{\phi_1\phi_2}$. This endows $\mu_M^n \rtimes \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ with a natural structure of semidirect product, given by

$$(\boldsymbol{g}_1, \phi_1) \cdot (\boldsymbol{g}_2, \phi_2) = (\phi_2^{-1}(\boldsymbol{g}_1)\boldsymbol{g}_2, \phi_1\phi_2).$$

By definition of $\delta_0(V)$, there exists a variety X such that V is an irreducible component of X and $\delta_0(V) = \delta(X)$. Let $G = \langle a \cdot b^{-1} \mid \ a, b \in T\rangle \subset \mu_M^n \rtimes \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$, and $S = \{(\boldsymbol{g}, \phi) \in G \mid \ \boldsymbol{g}V^{\phi}$ is imbedded in $X\}$. Notice that $(\phi(\boldsymbol{g}^{-1}), \phi^{-1})$ is the inverse of $(\boldsymbol{g}, \phi) \in \mu_M^n \rtimes \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$. Consider

$$\widetilde{X} = X \cap \Big(\bigcap_{(\boldsymbol{g},\phi)\in S} \phi(\boldsymbol{g}^{-1})X^{\phi^{-1}}\Big).$$

We have that V is an irreducible component of $\widetilde{X}$ and $\delta(\widetilde{X}) = \delta(X) = \delta_0(V)$. Moreover, no $\boldsymbol{g}V^{\phi}$ is imbedded in $\widetilde{X}$, for $(\boldsymbol{g}, \phi) \in G$. Assume by contradiction that there is a $\boldsymbol{g}V^{\phi}$ imbedded in $\widetilde{X}$. Since $\widetilde{X} \subset X$, $\boldsymbol{g}V^{\phi}$ is imbedded in X and so $(\boldsymbol{g}, \phi) \in S$. By induction, we suppose $(\boldsymbol{g}_n, \phi^n) = (\boldsymbol{g}, \phi)^n \in S$ for some $n \geq 1$. Then $\widetilde{X} \subset \phi^n(\boldsymbol{g}_n^{-1})X^{\phi^{-n}}$ and so $\boldsymbol{g}V^{\phi}$ is imbedded in $\phi^n(\boldsymbol{g}_n^{-1})X^{\phi^{-n}}$; which implies $(\boldsymbol{g}_{n+1}, \phi^{n+1}) = (\boldsymbol{g}, \phi)^{n+1} \in S$. Therefore, $(\boldsymbol{g}, \phi)^n \in S$ for every $n \in \mathbb{N}_{>0}$. In particular, taking $n = \mathrm{ord}((\boldsymbol{g}, \phi))$ we have that $(\boldsymbol{1}, \mathrm{Id}) \in S$, which is a contradiction.

Next we define

$$Y = \bigcup_{(\boldsymbol{g},\phi)\in T} \boldsymbol{g}\widetilde{X}^{\phi}.$$

Then $\bigcup_{(\boldsymbol{g},\phi)} \boldsymbol{g}V^{\phi} \subset Y$ and $\delta(Y) \leq t\delta(\widetilde{X}) = t\delta_0(V)$ by Lemma 1.2.9. Moreover, no $\boldsymbol{g}V^{\phi}$ is imbedded in Y, for $(\boldsymbol{g}, \phi) \in T$. Assume by contradiction that there is a $(\boldsymbol{g}, \phi) \in T$ such that $\boldsymbol{g}V^{\phi}$ is imbedded in Y. Then, there exists some $(\boldsymbol{g}_0, \phi_0) \in T$ such that $\boldsymbol{g}V^{\phi}$ is imbedded in $\boldsymbol{g}_0\widetilde{X}^{\phi_0}$. Thus $\phi_0(\boldsymbol{g}_0^{-1})(\boldsymbol{g}V^{\phi})^{\phi_0^{-1}} = \phi_0(\boldsymbol{g}_0^{-1}\boldsymbol{g})V^{\phi_0^{-1}\phi}$ is imbedded in $\widetilde{X}$ and, since $(\boldsymbol{g}, \phi) \cdot (\boldsymbol{g}_0, \phi_0)^{-1} = (\phi_0(\boldsymbol{g}_0^{-1}\boldsymbol{g}), \phi_0^{-1}\phi) \in G$, this contradicts the definition of $\widetilde{X}$. □

Remark 1.2.11. It is possible to give a slightly more general version of this statement, taking T a finite subset in $\mu_\infty^n \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The proof follows as the one of Lemma 1.2.10 after setting M to be the smallest integer satisfying that every element $(\boldsymbol{g}, \phi) \in T$ is such that $\boldsymbol{g} \in \mu_M^n$, and $(\mathbb{Q}^{\mathrm{ab}})^{\phi} \subset \mathbb{Q}(\zeta_M)$. This generalization is not needed in our application of Lemma 1.2.10.

The following lemma is a key ingredient in the proof of Theorem 1.2.16 for varieties defined over abelian extensions of $\mathbb{Q}$.

Let the closure of V in $\mathbb{P}^n$ be defined by the homogeneous radical ideal I in $\mathbb{C}[x_0, \ldots, x_n]$. For $\nu \in \mathbb{N}$, denote by $H(V; \nu)$ the *Hilbert function* $\dim(\mathbb{C}[x_0, \ldots, x_n]/I)_\nu$. Notice that if V is defined over $\mathbb{K}$, also is I as shown in the proof of Lemma 1.2.8. Hence, for $\nu \in \mathbb{N}$, one can define the Hilbert function $H(V; \nu)$ as $\dim(\mathbb{K}[x_0, \ldots, x_n]/I)_\nu$, since this value is invariant by base change.

The following sharp upper bound for the Hilbert function is a theorem of Chardin [23].

Theorem 1.2.12. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an equidimensional variety of dimension $d = n - k$, and let $\nu \in \mathbb{N}$. Then*

$$H(V; \nu) \leq \binom{\nu + d}{d} \deg(V).$$

On the other hand, as a consequence of a result of Chardin and Phillipon [24, Corollaire 3] on Castelnuovo's regularity, we have the following lower bound for the Hilbert function.

Theorem 1.2.13. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an equidimensional variety of dimension $d = n - k$, and $m = k(\delta_0(V) - 1)$. Then, for any integer $\nu > m$, we have*

$$H(V;\nu) \geq \binom{\nu + d - m}{d} \deg(V).$$

By means of these bounds, we aim to infer from Propositions 1.2.4 and 1.2.6 a hypersurface Z of degree $\delta_0(V)$ up to a multiplicative factor depending only on n and the dimension of V, such that $\overline{V_{\mathrm{tors}}} \subset V \cap Z \subsetneq V$. We first present the following intermediate result which we use for varieties defined over abelian extensions of $\mathbb{Q}$.

Lemma 1.2.14. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of dimension $d = n - k$. Let $M > 2$ be a positive integer, and fix ζ_M a primitive M-th root of unity, such that V is defined over $\mathbb{Q}(\zeta_M)$. Let $\phi \in \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ and let $\boldsymbol{\eta} \in \mu_M^n$.*

(a) If $\boldsymbol{\eta} V^{\phi} \neq V$, then there exists a homogeneous polynomial $F \in \mathbb{Q}^{ab}[x_0, \ldots, x_n]$ of degree at most $2k(2d+1)\delta_0(V)$ such that $F \equiv 0$ on $\boldsymbol{\eta} V^{\phi}$ and $F \not\equiv 0$ on V.

(b) If $V \not\subset [2]^{-1}(\boldsymbol{\eta} V^{\phi})$, then there exists a homogeneous polynomial $G \in \mathbb{Q}^{ab}[x_0, \ldots, x_n]$ of degree at most $2^n k(2d+1)\delta_0(V)$ such that $G \equiv 0$ on $[2]^{-1}(\boldsymbol{\eta} V^{\phi})$ and $G \not\equiv 0$ on V.

Proof. The proof of both cases is similar; however we choose to discuss both of them for the subtleties.

(a) Since V is an irreducible variety, $\boldsymbol{\eta} V^{\phi}$ is also irreducible and of the same degree. By Theorem 1.2.12 we get, for any $\nu \in \mathbb{N}$,

$$H(\boldsymbol{\eta} V^{\phi};\nu) \leq \binom{\nu + d}{d} \deg(V).$$

On the other hand, let $V' = V \cup \boldsymbol{\eta} V^{\phi}$. This is a d-equidimensional variety of degree $2\deg(V)$. Thereby, using Theorem 1.2.13 we have, for any $\nu > m$,

$$H(V';\nu) \geq \binom{\nu + d - m}{d} 2\deg(V),$$

where $m = k(\delta_0(V') - 1)$. In particular, $m \leq 2k\delta_0(V)$ due to Lemma 1.2.10. Fixing $\nu = m(2d+1)$, we obtain the following inequalities

$$\binom{\nu+d}{d}\binom{\nu+d-m}{d}^{-1} = \frac{(2dm+m+d)!}{(2dm+d)!}\cdot\frac{(2dm)!}{(2dm+m)!} = \prod_{j=1}^{d}\frac{(v+j)}{(v-m+j)}$$
$$\leq \left(1+\frac{m}{\nu-m}\right)^d = \left(1+\frac{1}{2d}\right)^d \leq \mathrm{e}^{1/2} < 2.$$

Hence, we have $H(\boldsymbol{\eta}V^{\phi};\nu) < H(V';\nu)$.

This implies that there exists a homogeneous polynomial F of degree ν such that $F \equiv 0$ on $\boldsymbol{\eta}V^{\phi}$, and $F \not\equiv 0$ on $V' = \boldsymbol{\eta}V^{\phi} \cup V$. In particular $F \not\equiv 0$ on V. Moreover, $\deg(F) = \nu \leq 2k(2d+1)\delta_0(V)$. Notice that $\boldsymbol{\eta}V^{\phi}$ and V are defined over $\mathbb{Q}^{\mathrm{ab}}$, so one can choose F with coefficients in $\mathbb{Q}^{\mathrm{ab}}$. This proves (a).

(b) Let $W = [2]^{-1}(\boldsymbol{\eta}V^{\phi})$. This is a d-equidimensional variety of degree $2^k\deg(V)$. By Theorem 1.2.12 we get, for any $\nu \in \mathbb{N}$,

$$H(W;\nu) \leq \binom{\nu+d}{d} 2^k \deg(V).$$

On the other hand, consider $\boldsymbol{H} = [2]^{-1}\operatorname{Stab}(V) = \operatorname{Stab}([2]^{-1}(V))$, and let $W' = \bigcup_{\boldsymbol{\eta}\in\boldsymbol{H}} \boldsymbol{\eta}\cdot V$. In fact if $r = \operatorname{codim}_{\mathbb{G}_{\mathrm{m}}^n}(\operatorname{Stab}(V))$, taking φ as in Corollary 1.2.3, we have that $\boldsymbol{H}/\operatorname{Stab}(V) \simeq \varphi(H) = [2]^{-1}\operatorname{Stab}(\varphi(V)) = \mu_2^r$. This variety W' is also a d-equidimensional variety of degree $2^r\deg(V)$, and $k < r \leq n$. Thereby, using Theorem 1.2.13 we have, for any $\nu > m$,

$$H(W';\nu) \geq \binom{\nu+d-m}{d} 2^r \deg(V),$$

where $m = k(\delta_0(W')-1)$. In particular, $m \leq 2^n k\delta_0(V)$ due to Lemma 1.2.10. Fixing $\nu = m(2d+1)$, we obtain the following inequalities

$$\binom{\nu+d}{d}\binom{\nu+d-m}{d}^{-1} \leq \mathrm{e}^{1/2} < 2^{r-k}.$$

Hence, we have $H(W;\nu) < H(W';\nu)$.

This implies that there exists a homogeneous polynomial $\widetilde{G}$ of degree ν such that $\widetilde{G} \equiv 0$ on $W = [2]^{-1}(\boldsymbol{\eta}V^{\phi})$, and $\widetilde{G} \not\equiv 0$ on W'. In particular, there exists an $\boldsymbol{\eta}_0 \in \boldsymbol{H}$ such that $\widetilde{G} \not\equiv 0$ on $\boldsymbol{\eta}_0 V$. Notice also that since W and W' are defined over $\mathbb{Q}^{\mathrm{ab}}$, one can choose $\widetilde{G}$ to have coefficients in $\mathbb{Q}^{\mathrm{ab}}$.

Let $G(\boldsymbol{x}) = \widetilde{G}(\boldsymbol{\eta}_0\cdot\boldsymbol{x}) \in \mathbb{Q}^{\mathrm{ab}}[x_0,\ldots,x_n]$. We have that $G \equiv 0$ on $\boldsymbol{\eta}_0^{-1}[2]^{-1}(\boldsymbol{\eta}V^{\phi})$. Since the stabilizer is an algebraic subgroup, we have

$$\operatorname{Stab}([2]^{-1}(\boldsymbol{\eta}V^{\phi})) = [2]^{-1}\operatorname{Stab}(\boldsymbol{\eta}V^{\phi}) = [2]^{-1}\operatorname{Stab}(V^{\phi}) = [2]^{-1}\operatorname{Stab}(V) = \boldsymbol{H}.$$

In particular, $\boldsymbol{\eta}_0^{-1} \in \operatorname{Stab}([2]^{-1}\boldsymbol{\eta}V^{\phi})$. So $G \equiv 0$ on $[2]^{-1}(\boldsymbol{\eta}V)$. In addition, $G \not\equiv 0$ on $\boldsymbol{\eta}^{-1}\boldsymbol{\eta}V = V$. Moreover, $\deg(G) = \nu \leq 2^n k(2d+1)$, which proves (b).

□

Notice that the cases of this result cover all the irreducible components of the varieties V' arising from Proposition 1.2.6. Since we use Lemma 1.2.10 in the proof, technically it does not include the variety of Proposition 1.2.4. We state the following lemma to cover also this case.

Lemma 1.2.15. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of dimension $d = n - k$, defined over a finite Galois extension $\mathbb{K}$ of $\mathbb{Q}$. Let $\phi \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ such that $V^\phi \neq V$. Then there exists a homogeneous polynomial $F \in \mathbb{K}[x_0, \dots, x_n]$ of degree at most $2k(d+1)\delta_0(V)$ such that $F \equiv 0$ on V^ϕ and $F \not\equiv 0$ in V.*

Proof. First of all, we prove that $\delta_0(V \cup V^\phi) \leq 2\delta_0(V)$, following the same inductive argument as in Lemma 1.2.10. By the definition of δ_0, there exists a variety X such that V is an irreducible component of X and $\delta(X) = \delta_0(V)$. Let $S = \{\psi \in \langle\phi\rangle \mid V^\psi$ is imbedded in $X\}$ and consider

$$\widetilde{X} = X \cap \bigcap_{\psi \in S} X^{\psi^{-1}}.$$

We have that V is an irreducible component of $\widetilde{X}$ and $\delta(\widetilde{X}) = \delta(X) = \delta_0(V)$. Moreover V^ψ is not imbedded in $\widetilde{X}$, for $\psi \in \langle\phi\rangle$, by the same inductive argument as in Lemma 1.2.10. If there was a $\phi \in \langle\phi\rangle$ such that V^ψ is imbedded in $\widetilde{X} \subset X$, this would imply that V^ψ is imbedded in X and so $\psi \in S$. By induction, if $\psi^n \in S$ for some $n \geq 1$, $\widetilde{X} \subset X^{\psi^{-n}}$. Hence V^ψ is imbedded in $X^{\psi^{-n}}$, and so $\psi^{n+1} \in S$. Therefore $\langle\phi\rangle = S$; in particular $\mathrm{Id} \in S$, which is a contradiction.

Next, for

$$Y = \widetilde{X} \cup \widetilde{X}^\phi,$$

we have that V and V^ϕ are irreducible components of Y and $\delta(Y) = 2\delta_0(V)$. Hence $\delta_0(V \cup V^\phi) \leq 2\delta_0(V)$.

The proof of the existence of a polynomial as in the statement is as the one of Lemma 1.2.14(a). □

The following theorem may be considered as a specialization of [2, Theorem 2.1] to torsion subvarieties.

Theorem 1.2.16. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of dimension $d = n - k > 0$, defined over $\overline{\mathbb{Q}}$, such that $\overline{V_{tors}} \neq V$. Let*

$$\theta_0 = \theta_0(V) = k(2^{2n} + 2^{n+1} - 2)(2d+1)\delta_0(V).$$

Then $\overline{V_{tors}}$ is contained in a hypersurface Z defined over $\overline{\mathbb{Q}}$ of degree at most θ_0, which does not contain V; that is $\overline{V_{tors}} \subset V \cap Z \subsetneq V$.

Proof. Let $\mathbb{K}$ be the field of definition of V. When $\mathbb{K}$ is an abelian extension of $\mathbb{Q}$, we may distinguish both cases arising in Proposition 1.2.6. Let N be as in (1.2.2). Since $(\boldsymbol{\xi} \cdot V)_{\mathrm{tors}} = \boldsymbol{\xi} \cdot V_{\mathrm{tors}}$, after possibly translating the hypersurface Z by $\boldsymbol{\xi}^{-1}$, we can assume that $N = N_{\mathbb{K}}$.

1. If $2 \nmid N$, by Proposition 1.2.6(1) we have that

$$\overline{V_{\mathrm{tors}}} \subset V' = \bigcup_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} \varphi^{-1}(\boldsymbol{\eta}) V \cup \bigcup_{\boldsymbol{\eta} \in \mu_2^r} [2]^{-1}(\varphi^{-1}(\boldsymbol{\eta}) V^{\sigma}),$$

   where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ maps $\zeta_N \mapsto \zeta_N^2$; and $V' \cap V \subsetneq V$.

   For each $\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}$ we have that $V \neq \varphi^{-1}(\boldsymbol{\eta})V$. By Lemma 1.2.14(a), we obtain a hypersurface $Z_{\boldsymbol{\eta}}$ defined over $\mathbb{Q}^{\mathrm{ab}}$ of degree at most $2k(2d+1)\delta_0(V)$ such that $\varphi^{-1}(\boldsymbol{\eta})V \subset Z_{\boldsymbol{\eta}}$, and $V \not\subset Z_{\boldsymbol{\eta}}$. Moreover, for each $\boldsymbol{\eta} \in \mu_2^r$, we also have that $V \not\subset [2]^{-1}(\varphi^{-1}(\boldsymbol{\eta})V^{\sigma})$, since $V \cap V' \subsetneq V$. Thus, by Lemma 1.2.14(b), we obtain a hypersurface $Z'_{\boldsymbol{\eta}}$ defined over $\mathbb{Q}^{\mathrm{ab}}$ of degree at most $2^n k(2d+1)\delta_0(V)$ such that $[2]^{-1}(\varphi^{-1}(\boldsymbol{\eta})V^{\sigma}) \subset Z'_{\boldsymbol{\eta}}$, and $V \not\subset Z'_{\boldsymbol{\eta}}$. For the union of these hypersurfaces

$$Z = \bigcup_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} Z_{\boldsymbol{\eta}} \cup \bigcup_{\boldsymbol{\eta} \in \mu_2^r} Z'_{\boldsymbol{\eta}},$$

   we have then

$$\deg(Z) \leq \sum_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} 2k(2d+1)\delta_0(V) + \sum_{\boldsymbol{\eta} \in \mu_2^r} 2^n k(2d+1)\delta_0(V) \leq \theta_0$$

   and $\overline{V_{\mathrm{tors}}} \subset V \cap V' \subset V \cap Z \subsetneq V$.

2. If $4 \mid N$, by Proposition 1.2.6(2) we have that

$$\overline{V_{\mathrm{tors}}} \subset V' = \bigcup_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} \varphi^{-1}(\boldsymbol{\eta}) V \cup \bigcup_{\boldsymbol{\eta} \in \mu_2^r} \varphi^{-1}(\boldsymbol{\eta}) V^{\tau},$$

   where $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ maps $\zeta_N \mapsto \zeta_N^{1+2N'}$; and $V' \cap V \subsetneq V$.

   We proceed as in (1), and by using Lemma 1.2.14(a) for each irreducible component of V', we obtain a hypersurface Z defined over $\mathbb{Q}^{\mathrm{ab}}$ such that

$$\deg(Z) \leq \sum_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} 2k(2d+1)\delta_0(V) + \sum_{\boldsymbol{\eta} \in \mu_2^r} 2k(2d+1)\delta_0(V) \leq \delta_0,$$

   and $\overline{V_{\mathrm{tors}}} \subset V \cap V' \subset V \cap Z \subsetneq V$.

Whenever $\mathbb{K} \not\subset \mathbb{Q}^{\mathrm{ab}}$, by Proposition 1.2.4 we have that $\overline{V_{\mathrm{tors}}} \subset V \cap V^{\varsigma} \subsetneq V$, for any non-trivial $\varsigma \in \mathrm{Gal}(\mathbb{K}/(\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{K}))$. Since $V \neq V^{\varsigma}$, by Lemma 1.2.15 there is a hypersurface Z defined over $\mathbb{K}$ of degree at most $2k(2d+1)\delta_0(V) \leq \theta_0$ such that $\overline{V_{\mathrm{tors}}} \subset V \cap Z \subsetneq V$. This concludes the proof. $\square$

Notice that this should not be used in the case of treating curves, since the direct approach yields already an optimal bound, see (1.2.3). This theorem proves useful in treating varieties of higher dimension where an iterative application of Bézout's theorem only leads a bound with an exponential exponent for $\deg(V)$.

1.2.3 Induction theorems

In this section we present the first main result of this chapter. Both of the proofs we give in this section follow the same lines as the ones of Theorems 2.2 and 1.2 in [2].

First, we state a theorem which serves as an intermediate result.

Theorem 1.2.17. *Let $V_0 \subset V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$, such that V_0 is irreducible, and V_1 is defined over $\overline{\mathbb{Q}}$. Let* $\operatorname{codim}(V_i) = k_i$, $i = 0, 1$. *Then, if $V_0 \not\subset \overline{V_{1,\mathrm{tors}}}$, there exists a hypersurface $Z \subset \mathbb{G}_{\mathrm{m}}^n$ defined over $\overline{\mathbb{Q}}$ of degree at most θ such that $V_0 \nsubseteq Z$ and $\overline{V_{0,\mathrm{tors}}} \subseteq Z$, where*

$$\theta = ((2n-1)k_0(2^{2n} + 2^{n+1} - 2))^{k_0 - k_1 + 1}\delta(V_1).$$

Proof. Assume that the statement in the theorem is false; that is, if Z is a hypersurface defined over $\overline{\mathbb{Q}}$ of degree at most θ containing $\overline{V_{0,\mathrm{tors}}}$, then it contains the whole variety V_0. We proceed by building a chain of varieties

$$X_{k_1} = V_1 \supseteq \cdots \supseteq X_{k_0+1}$$

satisfying, for every $r = k_1, \ldots, k_0 + 1$, the following:

(i) $V_0 \subset X_r$,

(ii) each irreducible component of X_r containing V_0 has at least codimension r.

If a such chain exists, then there is an irreducible component of X_{k_0+1} which is at least of codimension $k_0 + 1$ containing V_0. This yields a contradiction since the codimension of V_0 is k_0, and concludes the proof.

We construct a chain like this by recursion. We demand X_r to satisfy the following additional property for each X_r, $r = k_1, \ldots, k_0 + 1$:

(iii) $\delta(X_r) \leq D_r$,

where

$$D_r = (k_0(2^{2n} + 2^{n+1} - 2)(2n-1))^{r-k_1}\delta(V_1).$$

First, notice that for $r = k_1$ we already have that for the variety X_{k_1} properties (i)–(iii) hold.

Next, let us assume that for $r \geq k_1$ we have constructed the variety X_r in the chain, and write $X_r = W_1 \cup \cdots \cup W_t$ where the W_j's are the irreducible components of X_r.

After possibly renumbering, by (i) there exists an $s \geq 1$ such that $V_0 \subset W_j$ if and only if $1 \leq j \leq s$. By the hypothesis of the theorem, $V_0 \not\subset \overline{V_{1,\mathrm{tors}}}$, no W_j can be a torsion coset for $j = 1, \ldots, s$. Moreover, for these j's, we have $\operatorname{codim}(W_j) \leq k_0$ since $V_0 \subset W_j$, and $\delta_0(W_j) \leq \delta(X_r)$. Thus, for every $j = 1, \ldots, s$, Theorem 1.2.16 gives a hypersurface Z_j defined over $\overline{\mathbb{Q}}$ such that

$$\deg(Z_j) \leq k_0(2^{2n} + 2^{n+1} - 2)(2n-1)\delta(X_r) \leq D_{r+1},$$

and

$$\overline{W_{j,\mathrm{tors}}} \subset W_j \cap Z_j \subsetneq W_j. \tag{1.2.4}$$

The inclusion $V_0 \subset W_j$ also gives an inclusion of their respective torsion subvarieties. Hence Z_j is a hypersurface of degree at most $D_{r+1} \leq \theta$ containing $\overline{V_{0,\mathrm{tors}}}$. By the assumption in the proof, this implies that $V_0 \subset Z_j$.

With these Z_j's, we define

$$X_{r+1} = X_r \cap \bigcap_{j=1,\ldots,s} Z_j,$$

which is defined over $\overline{\mathbb{Q}}$. Since $V_0 \subset Z_j$, for all $j = 1, \ldots, s$, we have that $V_0 \subset X_{r+1}$, satisfying therefore property (i). To show that property (ii) holds for X_{r+1}, first observe that the only irreducible components of X_{r+1} containing V_0 are irreducible components of $W_j \cap Z_1 \cap \cdots \cap Z_s$ for every $j \leq s$. By construction of X_r we have that $\operatorname{codim}(W_j) \geq r$ for $j \leq s$, since $V_0 \subset W_j$ for these j's. Therefore, the second inclusion in (1.2.4) gives $\operatorname{codim}(W_j \cap Z_j) \geq r + 1$, and so item (ii) is satisfied for $r + 1$. Finally, property (iii) comes from the following inequalities

$$\delta(X_{r+1}) \leq \max\{\delta(X_r), \deg(Z_1), \ldots, \deg(Z_s)\} \leq D_{r+1}.$$ □

Theorem 1.2.18. *Let $V \subset \mathbb{G}_m^n$ be a variety of dimension $d > 0$. For $j = 0, \ldots, d$, let V^j_{tors} denote the j-equidimensional part of $\overline{V_{tors}}$. Then, for every $j = 0, \ldots, d$,*

$$\deg(V^j_{tors}) \leq c_{n,j}\delta(V)^{n-j},$$

where

$$c_{n,j} = ((2n-1)(n-1)(2^{2n} + 2^{n+1} - 2))^{d(n-j)}.$$

Proof. First, assume that V is defined over $\overline{\mathbb{Q}}$. Write $V = X^0 \cup \cdots \cup X^d$, where X^j are the j-equidimensional part of V, for $j = 0, \ldots, d$. For simplicity of notation, let us fix

$$\theta = ((2n-1)(n-1)(2^{2n} + 2^{n+1} - 2))^d \, \delta(V).$$

The key element is to prove the following inequality

$$\sum_{j=0}^{d} \theta^j \deg(V^j_{\mathrm{tors}}) \leq \sum_{j=0}^{d} \theta^j \deg(X^j). \tag{1.2.5}$$

We then apply a result of Philippon [66, Corollaire 5] as we detail next. With the notation as it appears *loc. cit.*, we take $m = n$, $S = \mathbb{P}^n$, $\delta = \theta$ and $Z_1, \dots, Z_l$ hypersurfaces of degree at most $\delta(V) \leq \theta$ that define V. By the definition of d_φ in [66, p. 347], when we apply Corollaire 5 in *loc. cit.* to $S_l = \mathbb{P}^n \cdot Z_1 \cdots Z_l$, we obtain

$$\sum_{j=0}^{d} \theta^j \deg(X^j) \leq \theta^n.$$

From this inequality follows, for every $j = 0, \dots, d$,

$$\deg(V^j_{\mathrm{tors}}) \leq \theta^{n-j} = c_{n,j}\, \delta(V)^{n-j},$$

proving the theorem.

The strategy to show inequality (1.2.5) is to build inductively a family of varieties $Y^d, \dots, Y^0$ satisfying, for each $r = d, \dots, 0$, the following:

(i) Y^r is r-equidimensional,

(ii) $\overline{V_{\mathrm{tors}}} \subseteq V^d_{\mathrm{tors}} \cup \cdots \cup V^{r+1}_{\mathrm{tors}} \cup Y^r \cup X^{r-1} \cup \cdots \cup X^0$,

(iii) $\sum_{j=r+1}^{d} \theta^{j-r} \deg(V^j_{\mathrm{tors}}) + \deg(Y^r) \leq \sum_{j=r}^{d} \theta^{j-r} \deg(X^j)$,

(iv) every irreducible component of Y^r intersects $\overline{V_{\mathrm{tors}}}$, and is not contained in $V^d_{\mathrm{tors}} \cup \cdots \cup V^{r+1}_{\mathrm{tors}}$.

Then inequality (1.2.5) is deduced by the inclusion $V^0_{\mathrm{tors}} \subset Y^0$, which gives $\deg(V^0_{\mathrm{tors}}) \leq \deg(Y^0)$.

Notice first that for $r = d$, X^d already satisfies (i)–(iii). We thus set Y^d to be the union of all irreducible components of X^d satisfying (iv). Next, let us assume that for $0 < r \leq d$ we already have a variety Y^r satisfying these properties and write

$$Y^r = V^r_{\mathrm{tors}} \cup W_1 \cup \cdots \cup W_s, \quad \text{for some } s \geq 0,$$

where the W_i's are the irreducible components of Y^r that are not in V^r_{tors}. Observe that if $s = 0$, X^{r-1} already satisfies (i)–(iii), so we may take Y^{r-1} to be the union of all irreducible components of X^{r-1} satisfying (iv). Hence we assume $s > 0$. Moreover, after possibly discarding some of these irreducible components, we can also assume that (iv) is satisfied. Hence, no W_i is included in a torsion coset of V.

For each $i = 1, \dots, s$, we apply Theorem 1.2.17 to $V_0 = W_i$ and $V_1 = V$, where $k_0 \leq n - 1$, which gives a hypersurface Z_i of degree at most θ such that $\overline{W_{i,\mathrm{tors}}} \subset W_i \cap Z_i \subsetneq W_i$. Then Krull's Hauptidealsatz implies that $W_i \cap Z_i$ is either empty or an $(r-1)$-equidimensional variety. We hence define

$$Y^{r-1} = X^{r-1} \cup \bigcup_{i=1,\dots,s} (W_i \cap Z_i).$$

By construction, Y^{r-1} verifies properties (i) and (ii) for $r-1$. Moreover, by Bézout's theorem we have

$$\deg(Y^{r-1}) \leq \theta \sum_{i=1}^{s} \deg(W_i) + \deg(X^{r-1}).$$

On the other hand, since $Y^r = V^r_{\mathrm{tors}} \cup W_1 \cup \cdots \cup W_s$, we may replace the inequality above by

$$\deg(Y^{r-1}) \leq \theta\big(\deg(Y^r) - \deg(V^r_{\mathrm{tors}})\big) + \deg(X^{r-1}).$$

The addition of $\sum_{j=r}^{d} \theta^{j+1-r} \deg(V^j_{\mathrm{tors}})$ on both sides of the inequality yields

$$\begin{aligned}\sum_{j=r}^{d} \theta^{j+1-r} \deg(V^j_{\mathrm{tors}}) + \deg(Y^{r-1}) \\ \leq \sum_{j=r}^{d} \theta^{j+1-r} \deg(V^j_{\mathrm{tors}}) + \theta\big(\deg(Y^r) - \deg(V^r_{\mathrm{tors}})\big) + \deg(X^{r-1}) \\ = \theta\left(\sum_{j=r+1}^{d} \theta^{j-r} \deg(V^j_{\mathrm{tors}}) + \deg(Y^r)\right) + \deg(X^{r-1}).\end{aligned}$$

By property (iii) in the induction step for r, the sum can be bounded above, and therefore

$$\theta\left(\sum_{j=r}^{d} \theta^{j-r} \deg(V^j_{\mathrm{tors}}) + \deg(Y^r)\right) + \deg(X^{r-1}) \leq \sum_{j=r-1}^{d} \theta^{j+1-r} \deg(X^j).$$

This shows that Y^{r-1} satisfies property (iii) for $r-1$, concluding the proof for V defined over $\overline{\mathbb{Q}}$.

To conclude the proof of the theorem, we have to deal with the case when V is not necessarily defined over $\overline{\mathbb{Q}}$. First we prove that if Z is a hypersurface defined over $\mathbb{C}$, and $Z' = \bigcap_{\phi \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})} Z^\phi$ where $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ denotes the automorphisms of $\mathbb{C}$ that fix the field $\overline{\mathbb{Q}}$, then Z' is a variety defined over $\overline{\mathbb{Q}}$. We do this in a similar fashion as Amoroso and Viada's proof of [3, Lemma 2.2].

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial whose zero set is $Z(f) = Z$, and write $f = \sum_{i=1}^{r} \lambda_i f_i$, where $f_1, \ldots, f_r \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ are linearly independent over $\overline{\mathbb{Q}}$. Notice that, for every $\phi \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, Z^ϕ is defined by the zeros of $f^\phi = \sum_{i=1}^{r} \phi(\lambda_i) f_i$, and in particular $Z^\phi \supset Z(f_1, \ldots, f_r)$. Hence

$$Z' \supset Z(f_1, \ldots, f_r). \tag{1.2.6}$$

Moreover, since $\overline{\mathbb{Q}}$ is a perfect field, by [12, Chapitre V, §15.6 Théorème 4(c)] there are $\phi_1, \ldots, \phi_r$ such that $\det\big(\phi_j(\lambda_i)\big)_{i,j} \neq 0$. So, for all $\boldsymbol{x} \in \mathbb{G}_{\mathrm{m}}^n$,

$$f^{\phi_j}(\boldsymbol{x}) = \sum_{i=1}^{r} \phi_j(\lambda_i) f_i(\boldsymbol{x}) = 0,\ \forall j = 1, \ldots, r \implies f_1(\boldsymbol{x}) = \cdots = f_r(\boldsymbol{x}) = 0.$$

Hence $Z' \subset \bigcap_{j=1}^{r} Z^{\phi_j} \subset Z(f_1, \ldots, f_r)$. Together with (1.2.6), this gives $Z' = Z(f_1, \ldots, f_r)$ and so Z' is defined over $\overline{\mathbb{Q}}$.

For $V \subset \mathbb{G}_{\mathrm{m}}^n$ a variety of any dimension, write V as the intersection of $Z_1, \ldots, Z_t$ hypersurfaces defined over $\mathbb{C}$. Notice that for every $\phi \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, $V^{\phi} = Z_1^{\phi} \cap \cdots \cap Z_t^{\phi}$. Then $V' := \bigcap_{\phi \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})} V^{\phi}$ is defined over $\overline{\mathbb{Q}}$, since $Z_j' := \bigcap_{\phi \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})} Z_j^{\phi}$ is defined over $\overline{\mathbb{Q}}$ and $V' = Z_1' \cap \cdots \cap Z_t'$.

In addition, $\overline{V_{\mathrm{tors}}}$ is defined over $\overline{\mathbb{Q}}$, so it is invariant by all the automorphisms in $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, and we have that $\overline{V_{\mathrm{tors}}} = \overline{V'_{\mathrm{tors}}}$. The statement of the theorem follows from the previous case over $\overline{\mathbb{Q}}$ applied to V' and the fact that $\delta(V') \leq \delta(V)$ and $\dim(V') \leq \dim(V)$. □

Remark. *Following the proofs of these theorems as presented by Amoroso and Viada [2] we obtain that $\delta_0(H) \leq \theta$, for each maximal torsion coset $\boldsymbol{\omega} \cdot H$ in V. Nevertheless, sharper bounds than this one are already known, for example [8, Theorem 3.3.8] gives*

$$\delta(H) \leq n\delta(V).$$

For a squarefree polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, the degree of f is equal to the degree of definition of the variety given by f. This gives the weak version of the conjecture in (1.1.2). Via homomorphisms one can deduce Ruppert's conjecture from Theorem 1.2.18.

Corollary 1.2.19. *Let $f \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ be a polynomial with $\deg_{x_i}(f) = d_i > 0$, for $i = 1, \ldots, n$, and V be the variety defined by the zeroes of f. Then the number of isolated torsion points in V is bounded above by*

$$c_{n,0}\, n^n\, d_1 \cdots d_n,$$

where

$$c_{n,0} = \left((2n-1)(n-1)(2^{2n} + 2^{n+1} - 2)\right)^{n^2-n}.$$

Proof. For each $j = 1, \ldots, n$, let $D_j = \frac{d_1 \ldots d_n}{d_j}$, and consider the homomorphism

$$\begin{aligned} [D_1, \ldots, D_n] : \mathbb{G}_{\mathrm{m}}^n &\longrightarrow \mathbb{G}_{\mathrm{m}}^n \\ (x_1, \ldots, x_n) &\longmapsto (x_1^{D_1}, \ldots, x_n^{D_n}), \end{aligned}$$

corresponding to the diagonal matrix with coefficients $D_1, \ldots, D_n$. The variety given by $f(x_1^{D_1}, \ldots, x_n^{D_n})$ is $W = [D_1, \ldots, D_n]^{-1} V$, and $\deg_{x_i}(W) = \prod_{i=1}^{n} d_i$, for every $i = 1, \ldots, n$. Then, W is of degree at most $n\, d_1 \cdots d_n$, and by applying Theorem 1.2.18 to W we obtain that

$$\# W^0_{\mathrm{tors}} \leq c_{n,0} (n\, d_1 \ldots d_n)^n.$$

The result follows from the fact that $\# W^0_{\mathrm{tors}} = \# \,\mathrm{Ker}([D_1, \ldots, D_n])\, \# V^0_{\mathrm{tors}}$. □

In Theorem 1.2.18 we could have given a more precise bound, depending on the field of definition of the variety V. To understand this, first observe that the varieties V' we obtain in Propositions 1.2.4 and 1.2.6 are defined over the same field as V. Hence, in Theorem 1.2.16 we could consider changing the definition of θ_0, depending on which field V is defined over. If the field of definition of V is an abelian extension of $\mathbb{Q}$, sharpening the value of θ_0 does not change significantly our bound because the order of n in the constants $c_{n,j}$'s remains essentially the same. However, in the case when V is not defined over $\mathbb{Q}^{\mathrm{ab}}$, Theorem 1.2.16 holds also for

$$\theta_0 = 2k(2d+1)\delta_0(V).$$

Using this definition of θ_0 in Theorems 1.2.17 and 1.2.18, we can improve the bound obtained in the latter. Hence, if V is not defined over $\mathbb{Q}^{\mathrm{ab}}$, the number of isolated torsion points in V can be bounded above by

$$\big(2(2n-1)(n-1)\big)^{n^2-n}\delta(V)^n.$$

1.2.4 Proof of the conjectures

The idea to prove Aliev-Smyth's conjecture is to proceed similarly as in the proof of Corollary 1.2.19. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial with Newton polytope Δ, and V be the hypersurface given by f. Our aim is to give a homomorphism $\varphi : \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$, such that the determinant of the matrix representing φ is equal to $\kappa_n \deg(\varphi^{-1}(V))^n$, where κ_n only depends on n. This direct approach does not work if we want to deal with any polytope. Instead, we consider a family of homomorphisms $\varphi_l : \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$ such that the limit

$$\lim_{l\to\infty} \deg(\varphi_l^{-1}(V))^n \det(\varphi_l)^{-1}$$

only depends on n.

First, we state a result of John [44, Theorem III] which allows us to compare the volume of any convex polytope Δ with the volume of the ellipsoid of smallest volume containing Δ.

Theorem 1.2.20. *Let $S \subset \mathbb{R}^n$ be a set such that its convex hull is of dimension n. If E is the ellipsoid of smallest volume containing S, then the ellipsoid E' which is concentric and homothetic to E at ratio $\frac{1}{n}$ is contained in the convex hull of S.*

An ellipsoid E in $\mathbb{R}^n$ is determined by an invertible matrix $M \in \mathrm{GL}_n(\mathbb{R})$ and a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$\boldsymbol{B}_n = M \cdot E + \boldsymbol{v} = \{M \cdot \boldsymbol{t} + \boldsymbol{v} \ | \ \boldsymbol{t} \in E\}, \tag{1.2.7}$$

where $\boldsymbol{B}_n$ represents the n-dimensional unit ball with respect to the L^2-norm, centered in $\mathbf{0}$. In particular, the volume of E is detemined by M:

$$\mathrm{vol}_n(E) = |\det(M)|^{-1}\omega_n,$$

where ω_n is the n-volume of $\boldsymbol{B}_n$.

For a polytope $\Delta \subset \mathbb{R}^n$ with integer vertices and of dimension n, John's result gives a way of including some affine deformation of Δ in a homothety of the standard simplex

$$\Delta^n = \{\boldsymbol{t} \in (\mathbb{R}_{\geq 0})^n \mid \ t_1 + \cdots + t_n \leq 1\}$$

in such a way that both volumes differ by a multiplicative factor depending only on n. The next proposition gives explicit construction of such translations and integer linear transformations.

Proposition 1.2.21. *Let $\Delta \subset \mathbb{R}^n$ be a convex polytope with integer vertices and of dimension n. For any $l \in \mathbb{N}_{>0}$, there exists a non-singular integer matrix $M_l \in \mathrm{GL}_n(\mathbb{Z})$ and an integer vector $\boldsymbol{\tau}_l$ such that*

$$M_l\Delta + \boldsymbol{\tau}_l \subset 2n(l + \sqrt{n}\,\mathrm{diam}_1(\Delta) + n)\Delta^n, \tag{1.2.8}$$

where $\mathrm{diam}_1(\Delta)$ represents the diameter of Δ with respect to the L^1-norm. Moreover, a family of such pairs $\{(M_l, \boldsymbol{\tau}_l)\}_{l>0}$ can be taken so that

$$\lim_{l\to+\infty} l^n|\det(M_l)|^{-1} \leq n^n\omega_n^{-1}\,\mathrm{vol}_n(\Delta). \tag{1.2.9}$$

Proof. After possibly translating Δ by an integer vector, we can always assume that

$$\Delta \subset (\mathbb{R}_{\geq 0})^n, \quad \text{and} \quad \Delta \cap \{\boldsymbol{t} \in \mathbb{Z}^n \mid t_i = 0\} \neq \emptyset, \text{ for every } i = 1, \dots, n.$$

Thus for any matrix $N \in \mathcal{M}_{n\times n}(\mathbb{R})$ with maximum norm $||N|| \leq 1$, we have

$$N\,\Delta \subset \sqrt{n}\,\mathrm{diam}_1(\Delta)\boldsymbol{B}_n. \tag{1.2.10}$$

Let E be the ellipsoid of smallest volume containing Δ, and $M \in \mathrm{GL}(\mathbb{R})$ and $\boldsymbol{v} \in \mathbb{R}^n$ be as in (1.2.7).

Next, choose $M_l \in \mathrm{GL}(\mathbb{Z})$ and $\boldsymbol{v}_l \in \mathbb{Z}^n$ to be integer approximations of lM and $l\boldsymbol{v}$ in the following sense:

$$M_l = lM + M', \quad ||M'|| < 1;$$

$$\boldsymbol{v}_l = l\boldsymbol{v} + \boldsymbol{v}', \quad ||\boldsymbol{v}'|| < 1;$$

where $||\cdot||$ denote the respective maximum norms.

Notice that, by inclusion (1.2.10) and the choice of matrices and vectors, we have

$$M_l\Delta + \boldsymbol{v}_l \subset l(M\cdot E + \boldsymbol{v}) + M'\Delta + \boldsymbol{v}' \subset l\boldsymbol{B}_n + \sqrt{n}\,\mathrm{diam}_1(\Delta)\boldsymbol{B}_n + n\boldsymbol{B}_n.$$

Thus, translating by $(l + \sqrt{n}\,\mathrm{diam}_1(\Delta) + n)\mathbf{1}$, we guarantee that the above convex bodies are all included in $(\mathbb{R}_{\geq 0})^n$. Therefore, taking

$$\boldsymbol{\tau}_l = (l + \sqrt{n}\,\mathrm{diam}_1(\Delta) + n)\mathbf{1} + \boldsymbol{v}_l,$$

we obtain

$$M_l\Delta + \boldsymbol{\tau}_l \subset (l + \sqrt{n}\,\mathrm{diam}_1(\Delta) + n)B_n + (l + \sqrt{n}\,\mathrm{diam}_1(\Delta) + n)\mathbf{1} \\ \subset 2n(l + \sqrt{n}\,\mathrm{diam}_1(\Delta) + n)\Delta^n.$$

It remains to prove that the inequation (1.2.9) holds for these M_l's and $\boldsymbol{\tau}_l$'s. Using John's result (Theorem 1.2.20), we have that $E' \subset \Delta$, where E' is an ellipsoid that is concentric and homothetic to E with ratio $\frac{1}{n}$. In particular,

$$\mathrm{vol}_n(E') = n^{-n}\,\mathrm{vol}_n(E) \quad \text{and} \quad \mathrm{vol}_n(E') \leq \mathrm{vol}_n(\Delta).$$

Therefore

$$|\det(M)|^{-1} = \omega_n^{-1}\,\mathrm{vol}_n(E) \leq \omega_n^{-1}\, n^n\,\mathrm{vol}_n(\Delta).$$

In addition, by our choice of M_l, we have that

$$\lim_{l\to+\infty} l^n |\det(M_l)|^{-1} = |\det(M)|^{-1}.$$

Inequality (1.2.9) follows then directly. □

By means of this proposition, we can take the bound in Theorem 1.2.18 and prove the conjecture of Aliev and Smyth. Before that, let us define the notion of degree related to a convex polytope we use in the theorem (see also Definition 2.2.4 for an equivalent definition).

Definition 1.2.22. Let $\Delta \subset \mathbb{R}^n$ be a convex polytope with integral vertices. Given a variety $V \subset \mathbb{G}_{\mathrm{m}}^n$ of dimension d, we define the degree associated to Δ as

$$\deg_\Delta(V) = \#(V \cap Z(f_1,\dots,f_d)),$$

where $f_1,\dots,f_d \in \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ are generic Laurent polynomials of Newton polytope Δ, and $Z(f_1,\dots,f_d)$ is the d-codimensional variety in $\mathbb{G}_{\mathrm{m}}^n$ defined by them.

This definition coincides with the degree of V with respect to the toric divisor D associated to Δ. Then global sections of $\mathcal{O}(D)$ are related to Laurent polynomials with Newton polytope Δ, see (2.2.5) for the precise statement.

Notice that $\deg_\Delta = \deg_{\Delta+\boldsymbol{\lambda}}$ for every integer $\boldsymbol{\lambda} \in \mathbb{Z}^n$. Moreover, from the inclusion of polytopes $\Delta_1 \subset \Delta_2$, it follows that

$$\deg_{\Delta_1}(V) \leq \deg_{\Delta_2}(V). \tag{1.2.11}$$

In particular, since the usual degree corresponds to $\deg_{\Delta^n}$, if Δ contains the standard simplex, we have $\deg(V) \leq \deg_\Delta(V)$.

To deal with polytopes of dimension strictly lower than n, we have to consider a relative version of volume of the polytope instead of simply vol_n. For $\Delta \subset \mathbb{R}^n$ a convex polytope with integer vertices, not necessarily of dimension n, we consider $\Lambda(\Delta)$ the lattice obtained after saturating the integer span of $\{\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \mid \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \Delta \cap \mathbb{Z}^n\}$. Then, the relative volume of Δ, $\mathrm{vol}_{\Lambda(\Delta)}(\Delta)$, is the volume of Δ for the Haar measure on $\Lambda(\Delta) \otimes_{\mathbb{Z}} \mathbb{R}$ normalized such that $\Lambda(\Delta)$ has covolume 1.

Theorem 1.2.23. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a variety of dimension d. Let $\Delta \subset \mathbb{R}^n$ be a convex polytope such V can be defined by polynomials in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with support lying in Δ. For $j = 0, \dots, d$, let V_{tors}^j denote the j-equidimensional part of $\overline{V_{tors}}$. Then, for every $j = 0, \dots, d$,*

$$\deg_\Delta(V_{tors}^j) \leq \widetilde{c}_{n,j}\, \mathrm{vol}_{\Lambda(\Delta)}(\Delta),$$

where

$$\widetilde{c}_{n,j} = 2^n n^{2n} \omega_n^{-1}((2n-1)(n-1)(2^{2n} + 2^{n+1} - 2))^{d(n-j)},$$

and ω_n is the volume of the n-sphere.

Proof. If Δ is not of dimension n, we reduce to the case of full dimension as follows. Fix a basis of $\Lambda(\Delta)$, and complete it to a basis of $\mathbb{Z}^n$ such that the covolume of the basis of $\Lambda(\Delta)$ in $\Lambda(\Delta) \otimes \mathbb{R}$ coincides with the covolume of the extended basis in $\mathbb{R}^n$. Then we can extend Δ to a polytope $\widehat{\Delta} \subset \mathbb{R}^n$ of dimension n, by taking the Minkowski sum of Δ with the vectors of the base extension. In particular, $\widehat{\Delta}$ is a convex polytope with integer vertices and such that Δ is a facet of $\widehat{\Delta}$. By (1.2.11), this implies that

$$\deg_\Delta(W) \leq \deg_{\widehat{\Delta}}(W),$$

for any subvariety $W \subset \mathbb{G}_{\mathrm{m}}^n$. Moreover, since the base extension preserves the covolume of the respective bases, we have that $\mathrm{vol}_{\Lambda(\Delta)}(\Delta) = \mathrm{vol}_n \widehat{\Delta}$. Therefore, we can assume that Δ is of dimension n.

Let M_l and $\boldsymbol{\tau}_l$ be as in Proposition 1.2.21. Let $\varphi_l : \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$ be the algebraic group endomorphism defined by M_l, see (1.2.1). By the inclusion (1.2.8), for any polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with support $\mathrm{supp}(f) \subset \Delta$, we have

$$\mathrm{supp}\left(f(\varphi_l(\boldsymbol{x})) \cdot \boldsymbol{x}^{\boldsymbol{\tau}_l}\right) \subset 2n(l + \sqrt{n}\, \mathrm{diam}_1(\Delta) + n)\Delta^n.$$

So we have that $f(\varphi_l(\boldsymbol{x}) \cdot \boldsymbol{x}^{\boldsymbol{\tau}_l})$ is of degree at most $2n(l + \sqrt{n}\, \mathrm{diam}_1(\Delta) + n)$.

Write $W = \varphi_l^{-1}(V)$. We have that $\delta(W) \leq 2n(l + \sqrt{n}\, \mathrm{diam}_1(\Delta) + n)$. In addition, for every $j = 0, \dots, d$, we have that $W_{\mathrm{tors}}^j = \varphi_l^{-1}(V_{\mathrm{tors}}^j)$. Then, for a fixed j, by Theorem 1.2.18, we have the following inequality:

$$\deg(W_{\mathrm{tors}}^j) \leq c_{n,j} \big(2n(l + \sqrt{n}\, \mathrm{diam}_1(\Delta) + n)\big)^{n-j}. \tag{1.2.12}$$

We proceed to compare $\deg(W^j_{\mathrm{tors}})$ and $\deg_\Delta(V^j_{\mathrm{tors}})$. To do this, take generic Laurent polynomials $f_1,\dots,f_j$ with Newton polytope Δ, and so

$$\deg_\Delta(V^j_{\mathrm{tors}}) = \#(V^j_{\mathrm{tors}} \cap Z(f_1,\dots,f_j)).$$

Given a polynomial g, the zeroes of $g(\boldsymbol{x})$ and $g(\boldsymbol{x})\cdot \boldsymbol{x}^{\boldsymbol{\tau}_l}$ define the same variety. Hence

$$\varphi_l^{-1}(V^j_{\mathrm{tors}} \cap Z(f_1,\dots,f_j)) = W^j_{\mathrm{tors}} \cap Z\big(f_1(\varphi_l(\boldsymbol{x}))\cdot \boldsymbol{x}^{\boldsymbol{\tau}_l},\dots,f_j(\varphi_l(\boldsymbol{x}))\cdot \boldsymbol{x}^{\boldsymbol{\tau}_l}\big).$$

Then, Bézout's theorem gives

$$\#\Big(W^j_{\mathrm{tors}}\cap Z\big(f_1(\varphi_l(\boldsymbol{x}))\cdot \boldsymbol{x}^{\boldsymbol{\tau}_l},\dots,f_j(\varphi_l(\boldsymbol{x}))\cdot \boldsymbol{x}^{\boldsymbol{\tau}_l}\big)\Big) \le \deg(W^j_{\mathrm{tors}})\big(2n(l+\sqrt{n}\,\mathrm{diam}_1(\Delta)+n)\big)^j,$$

and since $\#(\varphi_l^{-1}(\boldsymbol{y})) = |\det(M_l)|$ for any point $\boldsymbol{y} \in \mathbb{G}_{\mathrm{m}}^n$, we have

$$\begin{aligned}|\det(M_l)|\deg_\Delta(V^j_{\mathrm{tors}}) &= \\ &\#\varphi_l^{-1}(V^j_{\mathrm{tors}} \cap Z(f_1,\dots,f_j)) \le \deg(W^j_{\mathrm{tors}})\big(2n(l+\sqrt{n}\,\mathrm{diam}_1(\Delta)+n)\big)^j.\end{aligned}$$

Combining this inequality with the one in (1.2.12), we obtain

$$\deg_\Delta(V^j_{\mathrm{tors}}) \le c_{n,j}(2n)^n\ (l+\sqrt{n}\,\mathrm{diam}_1(\Delta)+n)^n|\det(M_l)|^{-1}. \tag{1.2.13}$$

Finally, we use the inequality (1.2.9) and take the limit $l\to\infty$ in (1.2.13) to conclude

$$\deg_\Delta(V^j_{\mathrm{tors}}) \le c_{n,j}\ 2^n n^{2n}\omega_n^{-1}\,\mathrm{vol}_n(\Delta). \qquad \square$$

Notice that $\deg_\Delta$ of 0-dimensional varieties does not depend on the polytope. Therefore, equation (1.1.5) is a direct consequence of this theorem.

Remark. *Given $V \subset \mathbb{G}_{\mathrm{m}}^n$ a variety defined by dense polynomials, that is their Newton polytopes are homotheties of the standard simplex; we observe that the bound coming from Theorem 1.2.18 and the one from Theorem 1.2.23 differ only by a multiplying factor $2^n n^{2n}\omega_n$. This does not increase the order in n of the constants given by these theorems.*

Both conjectures follow as a direct consequence to this theorem. Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a hypersurface given by a polynomial $f \in \mathbb{C}[x_1,\dots,x_n]$. If we take $\Delta = [0,d_1]\times\cdots\times[0,d_n]$ where $(d_1,\dots,d_n)$ is the multidegree of f, Theorem 1.2.23 for $j=0$ proves Ruppert's conjecture (Conjecture 1.1.1). Even though a slightly better constant could be obtained directely from Theorem 1.2.18, see Corollary 1.2.19. On the other hand, if we take Δ as the Newton polytope of f, Theorem 1.2.23 for $j=0$ proves Aliev-Smyth's conjecture (Conjecture 1.1.2).

1.2.5 Example

We build an example to show that the dependence on the multidegree in Ruppert's conjecture (Conjecture 1.1.1) is optimal and the constant c_n must depend on n. To do this, we first present a result of Conway and Jones on vanishing sums of roots of unity. Let us define, for $m \in \mathbb{N}_{>0}$,

$$\Psi(m) := 2 + \sum_{\substack{p|m \\ p \text{ prime}}} (p-2).$$

The theorem of Conway and Jones [26, Theorem 5] states the following.

Theorem 1.2.24. *Let $\xi_1, \ldots, \xi_N$ be N roots of unity. Let $a_1, \ldots, a_N \in \mathbb{Z}$ such that $S = a_1\xi_1 + \ldots + a_N\xi_N = 0$ is minimal; that is there are no non-trivial vanishing subsums of S. Let*

$$m = \operatorname{lcm}(\operatorname{ord}(\xi_2/\xi_1), \ldots, \operatorname{ord}(\xi_N/\xi_1)).$$

Then $\Psi(m) \leq N$.

We present the following consequence to this result, which we use in the construction of our example.

Lemma 1.2.25. *Let $p_1, \ldots, p_n$ be n different primes such that $p_i > n+1$ for every $i = 1, \ldots, n$, and $\omega_1, \ldots, \omega_n$ be roots of unity such that*

$$S := \zeta_{p_1} + \cdots + \zeta_{p_n} + \omega_1 + \cdots + \omega_n = 0.$$

Then, up to reordering, $S = S_1 + \cdots + S_n$, where $S_i = \zeta_{p_i} + \omega_i = 0$, for every $i = 1, \ldots, n$.

Proof. Let $\zeta_{p_1} + \cdots + \zeta_{p_n} + \omega_1 + \cdots + \omega_n = S_1 + \cdots + S_t$, $t \geq 1$, be a decomposition in minimal vanishing and non-trivial subsums. We have to prove that each S_j contains at most one term ζ_{p_i}.

If this is not the case, there exists a minimal vanishing subsum S with at least three elements. Without loss of generality, we may assume that ζ_{p_1} and ζ_{p_2} are summands of S. Then taking m as in Theorem 1.2.24, we have that $p_1 \cdot p_2 | m$. Therefore

$$\Psi(m) \geq \Psi(p_1 \cdot p_2) = p_1 + p_2 - 2 > 2n.$$

On the other hand, by the minimality of S, Theorem 1.2.24 implies that $\Psi(m) \leq 2n$. This gives the contradiction that yields the proof. $\square$

Example 1.2.26. Let $p_1, \ldots, p_n$ be n different primes such that $p_i > n+1$, for every $i = 1, \ldots, n$. Let W be the variety defined by the zeros of

$$g(x_1, \ldots, x_n) = x_1 + \cdots + x_n - (\zeta_{p_1} + \cdots + \zeta_{p_n}).$$

By Lemma 1.2.25, we have

$$W_{\mathrm{tors}} = \Big\{ \boldsymbol{\omega} \in \mathbb{G}_{\mathrm{m}}^n \ \mid \ \{\omega_1, \dots, \omega_n\} = \{\zeta_{p_1}, \dots, \zeta_{p_n}\} \Big\}.$$

Thus, $\overline{W_{\mathrm{tors}}} = W_{\mathrm{tors}}$ is a finite set with $n!$ elements.

Let $d_1, \dots, d_n \in \mathbb{N}_{>0}$, and consider the homomorphism associated to the diagonal matrix $(d_1, \dots, d_n)$:

$$\begin{aligned} [d_1, \dots, d_n] : \mathbb{G}_{\mathrm{m}}^n &\longrightarrow \mathbb{G}_{\mathrm{m}}^n \\ \boldsymbol{x} &\longmapsto (x_1^{d_1}, \dots, x_n^{d_n}). \end{aligned}$$

Let $V = [d_1, \dots, d_n]^{-1}(W)$, which is the hypersurface in $\mathbb{G}_{\mathrm{m}}^n$ defined by the zeros of

$$f(x_1, \dots, x_n) = x_1^{d_1} + \dots + x_n^{d_n} - (\zeta_{p_1} + \dots + \zeta_{p_n}).$$

Then we have that the torsion subvariety of V is the preimage of $\overline{W_{\mathrm{tors}}}$, which is

$$\overline{V_{\mathrm{tors}}} = \Big\{ \boldsymbol{\omega} \in \mathbb{G}_{\mathrm{m}}^n \ \mid \ \{\omega_1^{d_1}, \dots, \omega_n^{d_n}\} = \{\zeta_{p_1}, \dots, \zeta_{p_n}\} \Big\}.$$

Which allows us to conclude that the number of (isolated) torsion points in V is $n! d_1 \cdots d_n$, proving the dependence on the multidegree of Corollary 1.2.19 to be optimal.

A further remark can be made for the bounds on the number of j-dimensional torsion cosets that follow from Theorem 1.2.18, for positive values of j.

Remark 1.2.27. Fix $j = 1, \dots, n-1$. Similar to the example above, we can construct a variety $W \subset \mathbb{G}_{\mathrm{m}}^{n-j}$ such that $W_{\mathrm{tors}} = W_{\mathrm{tors}}^0$ is a set of $(n-j)!$ elements. Consider the group homomorpshism

$$\varphi \colon \mathbb{G}_{\mathrm{m}}^n \twoheadrightarrow \mathbb{G}_{\mathrm{m}}^{n-j}, \quad (x_1, \dots, x_n) \longmapsto (x_1^d, \dots, x_{n-j}^d).$$

Then $V = \varphi^{-1}(W)$ is a variety with non trivial stabilizer, indeed $\mathrm{Stab}(V) \simeq \mathbb{G}_{\mathrm{m}}^j$, and

$$\overline{V_{\mathrm{tors}}} = \Big\{ \boldsymbol{\omega} \times \mathbb{G}_{\mathrm{m}}^j \subset \mathbb{G}_{\mathrm{m}}^{n-j} \times \mathbb{G}_{\mathrm{m}}^j \ \mid \ [d]\boldsymbol{\omega} \in W_{\mathrm{tors}} \Big\}.$$

This implies that $\overline{V_{\mathrm{tors}}}$ is the union of $(n-j)! d^{n-j}$ distinct j-dimensional torsion cosets, and shows also the optimality of the bound for positive dimensional torsion cosets in terms of the degree of the variety.

1.3 Bounds for the abelian Manin-Mumford

Let A be an abelian variety of dimension g defined over a number field K. After possibly replacing K by a finite algebraic extension, we assume that K satisfies that the l-adic representations attached to A are independent in the sense of Theorem 1.3.1, and all the simple factors of A are defined over K. Let $\iota : A \hookrightarrow \mathbb{P}^n$ be a fixed closed immersion into a projective space some dimension n, given by a very ample symmetric line bundle. Moreover, we assume that $i(A)$ is projectively normal subvariety of $\mathbb{P}^n$.

When considering subvarieties of A they are defined over a fixed algebraic closure of K unless stated otherwise. Moreover, when we say that a variety is irreducible, we imply it is geometrically irreducible.

This section is an analogy of the previous in the case of Abelian varieties, we may therefore choose to omit a complete exposition of some of the proofs due to its similarities to their toric analogues.

1.3.1 Galois action on torsion points

Although there seems to be a common behaviour between torsion points in the torus and in abelian varieties, the more complex structure of the latter ones also transpires in our setting.

First, to fix notations, we denote the *multiplication map by k*, with $k \geq 0$, as the isogeny

$$\begin{aligned} [k]\colon A &\longrightarrow A \\ P &\longmapsto \overbrace{P + \cdots + P}^{k \text{ times}}, \end{aligned}$$

whose kernel are the k-torsion points of A. The multiplication maps are defined by algebraic polynomials when we consider A as a subvariety of the projective space, which implies the algebraicity of torsion points.

As comparison with the previous section, a torsion point in the torus, being a vector of roots of unity, is always defined over a cyclotomic extension of $\mathbb{Q}$, namely the $\mathbb{Q}(\zeta_k)$ where k is the order of the torsion point and ζ_k a primitive k-th root of unity. Thus, the Galois action on torsion points in $\mathbb{G}_{\mathrm{m}}^n$ is a well understood topic. However, this is not the case for abelian varieties. The field of definition of a torsion point is not straightforwardly determined, strongly depending on the choice of A. Nevertheless, Galois automorphisms fixing the base field K do not change the order of a torsion point. This motivates the study of ℓ-adic representations attached to abelian varieties, which we briefly discuss below.

For a natural number k, the group of the k torsion points of A, denoted as $A[k]$, is a $\mathbb{Z}/k\mathbb{Z}$-module. Given a prime ℓ, we define the *ℓ-adic Tate module* of A as

$$\mathrm{T}_\ell(A) = \lim_{\substack{\longleftarrow \\ k}} A[\ell^k],$$

which is a free $\mathbb{Z}_\ell$-module of rank $2g$. The absolute Galois group of K, $\mathrm{Gal}(\overline{K}/K)$, acts over $\mathrm{T}_\ell(A)$ by a representation

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(\mathrm{T}_\ell(A)) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$$

which is induced by the Galois action on each $A[\ell^k]$, $k \geq 0$. For simplicity, we denote by $G_{K,\ell}$ the image by ρ_ℓ of the absolute Galois group of K. Bogomolov proved that $G_{K,\ell}$ contains an open subgroup of the homotheties $\mathbb{Z}_\ell^*$ of $\mathrm{GL}_{2g}(\mathbb{Z}_\ell)$, see [7, Théorème 2] and [6]. That is, for every prime ℓ, the index

$$\mathrm{c}_\ell = [\mathbb{Z}_\ell^* : G_{K,\ell} \cap \mathbb{Z}_\ell^*]$$

is finite. A long-standing conjecture of Lang states that $\mathrm{c}_\ell = 1$ for all but a finite number of primes ℓ. For elliptic curves without complex multiplication this is a consequence to Serre's open image theorem (see Théorème 3 and Corollaire of Théorème 5 in [79]). A further result of Serre states that c_ℓ can be bounded independently of ℓ for any Abelian variety.

The family of ℓ-adic representations defines a representation

$$\rho = \prod_\ell \rho_\ell \colon \mathrm{Gal}(\overline{K}/K) \longrightarrow \prod_\ell \mathrm{Aut}(\mathrm{T}_\ell(A)) \simeq \mathrm{GL}_{2g}(\hat{\mathbb{Z}}).$$

The following result of Serre [80, Théorème 1] gives a way of glueing all these together. We refer to [81, Théorème 1] for a proof of the statement.

Theorem 1.3.1. *There is a finite extension K' of K such that $\rho\colon \mathrm{Gal}(\overline{K'}/K') \to \prod_\ell G_{K',\ell}$ is surjective.*

Indeed this finite extension K' of K depends on A. Without loss of generality, we may replace K by K' in the sequel. This, together with the existence of a bound for the c_ℓ's for varying ℓ, gives the following result which is also due to Serre [80, Théorème 2']. We refer to [85, Théorème 3] for a proof of the statement.

Theorem 1.3.2. *There is an integer $\mathrm{c} \geq 1$ such that if n and k are coprime positive integers, there is a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$ satisfying*

$$\sigma_{|A[n]} = [k^{\mathrm{c}}]_{|A[n]}.$$

Remark 1.3.3. The problem of finding an explicit effective expression for the constant c (and K') in terms of A is still open. Some advancements towards obtaining an explicit constant where made by Wintenberger in [85], where he gives a proof of Theorem 1.3.2.

Giving an explicit value of c is also a simple instance of explicit versions of Mumford-Tate's conjecture on the closure of the whole $\rho(\mathrm{Gal}(\overline{K}/K))$. Recent results in this direction were made by Lombardo, who gave first explicit versions of Serre's open image theorem in [55, Theorem 9.1], and then extended to some more general cases. Nevertheless these results aim at a much more ambitious problem, and the bounds obtained do not seem suitable for our purpose of finding sharp bounds.

For a subvariety $V \subset A$, we define the *stabilizer* of V in A as

$$\mathrm{Stab}(V) = \{P \in A(\overline{K}) \mid P + V = V\}.$$

As it was the case for the torus, we have $\dim(\mathrm{Stab}(V)) \leq \dim(V)$. When V is irreducible, the equality holds if and only if V is a translate of an irreducible algebraic subgroup of A. By Poincaré's irreducibility theorem, the abelian variety A is isogenous to a product of abelian varieties

$$B \times \mathrm{Stab}(V)^0 \tag{1.3.1}$$

where $\mathrm{Stab}(V)^0$ is the connected component of $\mathrm{Stab}(V)$ containing 0, and B is an abelian subvariety of A. Then, by taking the quotient of A by $\mathrm{Stab}(V)$ we obtain an abelian variety which is isogenous to B. By abuse of notation, we denote by B the abelian variety obtained by this quotient. So, there exists a surjective group homomorphism

$$\varphi : A \longrightarrow\!\!\!\!\!\rightarrow B, \tag{1.3.2}$$

such that $\mathrm{Ker}(\varphi) = \mathrm{Stab}(V)$. In particular $\varphi(V)$ is a subvariety of B with trivial stabilizer. Up to replacing K by a finite extension, we can assume that all the simple factors of A are defined over K, and so is also φ. From here on forward, when we refer to the field K over which A is defined, we always assume that all the simple factors of A are also defined over K.

Let $V \subset A$ be a subvariety defined over $\overline{K}$. We denote by K_V the minimal algebraic extension of K such that V is defined over it, and then we say that V is *defined over* K_V. In particular, if φ is as in (1.3.2), $\varphi(V)$ is also defined over K_V.

The first case we need to consider is when K_V is not contained in $K(A_{\mathrm{tors}})$. Here $K(A_{\mathrm{tors}})$ plays the role of $\mathbb{Q}^{\mathrm{ab}}$ in Proposition 1.2.4, yielding by the same arguments the following result.

Proposition 1.3.4. *Let $V \subset A$ be an irreducible variety of positive dimension that is not defined over $K(A_{tors})$. For every non-trivial automorphism $\varsigma \in \mathrm{Gal}(K_V/(K_V \cap K(A_{tors})))$, we have*

$$V_{tors} \subset V \cap V^{\varsigma} \subsetneq V.$$

The rest of this section is devoted to the case when $K_V \subset K(A_{\mathrm{tors}})$. This case is more involved because of the fields of definition of torsion points in abelian varieties. Denote by v_2 the 2-*adic valuation* of an integer, and

$$\mathrm{c}_2 = v_2(\mathrm{c}),$$

with c the integer constant from Theorem 1.3.2. Fix $M \geq 1$ the smallest integer such that

$$K_V \subset K(A[M]) \quad \text{and} \quad v_2(M) \geq \mathrm{c}_2 + 2. \tag{1.3.3}$$

For every M-torsion point $R \in A[M]$, we consider the set $\mathcal{N}(R)$ of integers $\alpha > -v_2(M)$, such that there exists a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$ satisfying

$$\sigma_{|A[M]} = [(1 + 2^\alpha M)^{\mathrm{c}}]_{|A[M]} \quad \text{and} \quad (V + R)^\sigma = V + R.$$

Notice that $\alpha > -v_2(M)$ implies the coprimality of M and $1 + 2^\alpha M$. Henceforth, it enables the use of Theorem 1.3.2 to show in the first place the existence of a σ with a such restriction to $A[M]$.

Remark. *For every non-negative $\alpha \in \mathcal{N}(R)$, M and $1 + 2^\alpha M$ are coprime, by Theorem 1.3.2 there exists a $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that $\sigma_{|A[M]} = [(1 + 2^\alpha M)^{\mathrm{c}}]_{|A[M]}$. We have that $(1 + 2^\alpha M)^{\mathrm{c}} \equiv 1 \pmod{M}$. Thus, for each M-torsion point $R \in A[M]$ we have that $[(1 + 2^\alpha M)^{\mathrm{c}}]R = R$. Moreover, since $K_V \subset K(A[M])$, this implies that*

$$(V + R)^\sigma = V^\sigma + R^\sigma = V + R.$$

Hence, $\mathbb{N} \subset \mathcal{N}(R)$.

It then makes sense to take $\beta(R)$ to be the biggest integer in $\mathbb{Z} \setminus \mathcal{N}(R)$. Take

$$\beta = \min_{R \in A[M]} \beta(R). \tag{1.3.4}$$

In particular, we have $-v_2(M) < \beta \leq -1$.

Since $V_{\mathrm{tors}} + R = (V + R)_{\mathrm{tors}}$ for any torsion point R, throughout this paragraph we will continuously assume that $\beta = \beta(0)$. Then, we define

$$N = 2^{\beta+1} M. \tag{1.3.5}$$

It is an integer since $\beta \geq -v_2(M)$, and in fact even. This integer plays the same role of the integer N defined in (1.2.2) for the toric case.

Let us give an easy computation on the behaviour of the 2-adic valuation of the coefficients in binomial expansions.

Lemma 1.3.5. *Let $2 \leq \gamma \leq \delta$ be two integers. For any integer k with 2-adic valuation $v_2(k) \geq 2$, we have*

$$v_2\left(\binom{\delta}{\gamma} k^\gamma\right) \geq v_2(k) + v_2(\delta) + 1.$$

Proof. First, since $\gamma, \delta \geq 1$, we have

$$\binom{\delta}{\gamma} = \frac{\delta}{\gamma}\binom{\delta-1}{\gamma-1}.$$

Thus, by developing we obtain

$$v_2\left(\binom{\delta}{\gamma}k^{\gamma}\right) \geq v_2(\delta) - v_2(\gamma) + \gamma\, v_2(k) = v_2(\delta) + v_2(k) + (\gamma-1)v_2(k) - v_2(\gamma).$$

Since $v_2(k) \geq 2$, the proof can be reduced to the simple verification of

$$2\gamma - 2 - v_2(\gamma) \geq 1.$$

The statement follows then trivially by the choice of $\gamma \geq 2$. $\square$

This allows us to better bound the value β.

Lemma 1.3.6. *Let $V \subset A$, and β defined in* (1.3.4)*. Then*

$$\beta \leq -\mathrm{c}_2 - 1.$$

In particular, for M and N as in (1.3.3) *and* (1.3.5)*, respectively, we have*

$$v_2(N) + \mathrm{c}_2 \leq v_2(M).$$

Proof. The inequality is equivalent to showing that, for each $R \in A[M]$,

$$\beta(R) \leq -\mathrm{c}_2 - 1.$$

This is trivially true when $\mathrm{c}_2 = 0$.

Assume $\mathrm{c}_2 \geq 1$. Fix an integer $\alpha \geq -\mathrm{c}_2$, we develop the binomial

$$(1 + 2^{\alpha}M)^{\mathrm{c}} = 1 + 2^{\alpha}\,\mathrm{c}\,M + \sum_{\gamma=2}^{\mathrm{c}}\binom{\mathrm{c}}{\gamma}(2^{\alpha}\,M)^{\gamma}.$$

Notice that $2^{\alpha}\mathrm{c} \in \mathbb{N}$, because of the choice of α. Hence, M divides $2^{\alpha}\,\mathrm{c}\,M$. Moreover, by (1.3.3), we have that $v_2(2^{\alpha}M) \geq \alpha + \mathrm{c}_2 + 2 \geq 2$. Thus, Lemma 1.3.5 shows that, for every $\gamma \geq 2$,

$$v_2\Big(\binom{\mathrm{c}}{\gamma}(2^{\alpha}\,M)^{\gamma}\Big) \geq v_2(M) + \alpha + c_2 + 1 > v_2(M),$$

which gives that M divides $\binom{\mathrm{c}}{\gamma}(2^{\alpha}\,M)^{\gamma}$, for $\gamma \geq 2$. We then have

$$[(1 + 2^{\alpha}\,M)^{\mathrm{c}}]_{|A[M]} = \mathrm{Id}\,.$$

So $\beta(R) \leq -\mathrm{c}_2 - 1$, for every $R \in A[M]$, which concludes the proof. $\square$

With these tools we can give an explicit description of another subvariety containing V_{tors}.

Proposition 1.3.7. *Let $V \subset A$ be an irreducible subvariety of A defined over $K(A_{tors})$. Let M and β be the integers defined in* (1.3.3) *and* (1.3.4)*, and assume that β attains its minimum at* 0*. Then there exist two Galois automorphisms $\sigma, \rho \in \mathrm{Gal}(\overline{K}/K)$ whose respective restrictions to $A[M]$ are*

$$\sigma_{|A[M]} = [(2 + 2^{-v_2(M)}M)^{\mathrm{c}}]_{|A[M]} \quad \textit{and} \quad \rho_{|A[M]} = [(1 + 2^{\beta}M)^{\mathrm{c}}]_{|A[M]}, \tag{1.3.6}$$

such that

$$V' := \bigcup_{P \in A[4\mathrm{c}]} [2^{\mathrm{c}}]^{-1}(V^{\sigma} + P) \ \cup \bigcup_{P \in A[2]} (V^{\rho} + P) \ \cup \bigcup_{P \in A[2] \setminus \{0\}} (V + P))$$

satisfies $V_{\text{tors}} \subset V'$.

Proof. Fix a torsion point $Q \in V_{\text{tors}}$ of order $l \geq 1$.

The strategy of the proof starts by considering three different cases according to the 2-adic valuation of l. For each of these, we obtain a Galois automorphism whose action on $A[l]$ can be easily described. If $v_2(l) \leq \mathrm{c}_2 + 2$, we show that there is an element $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that $Q^{\sigma} \equiv [2^{\mathrm{c}}]Q \pmod{A[4c]}$. If $\mathrm{c}_2 + 2 < v_2(l) \leq \mathrm{c}_2 + \beta + 1 + v_2(M)$, we show that there is an element $\rho \in \mathrm{Gal}(\overline{K}/K)$ such that $Q^{\rho} \equiv Q \pmod{A[2]}$. If $v_2(l) > \mathrm{c}_2 + \beta + 1 + v_2(M)$, we show that there is an element $\tau \in \mathrm{Gal}(\overline{K}/K)$ such that $Q^{\tau} - Q \in A[2]$. Moreover, these automorphisms σ, ρ and τ can be chosen such that their restrictions to $A[M]$ are independent of Q and l; being this restriction as in (1.3.6) for σ and ρ, and $\tau_{|A[M]} = \mathrm{Id}$. In particular, the restriction to $K_V \subset K(A[M])$ does not depend on Q and l.

Before giving the details of the proof, we introduce the following notation:

$$m = \mathrm{lcm}(l, M), \quad \text{and} \quad m' = 2^{-v_2(m)}m.$$

To have a certain control on the p-adic difference of l and M, for $p > 2$, we denote by $\lambda, \mu \in \mathbb{Z}$ two coefficients satisfying the Bézout identity $(2^{v_2(M)})\lambda + \left(\frac{m'}{2^{-v_2(M)}M}\right)\mu = 1$. Then

$$m'\mu \equiv 2^{-v_2(M)}M \pmod{M} \tag{1.3.7}$$

The fact that $2 \nmid m'\mu$ follows from the definition of M in (1.3.3) and is strongly used below. It should be kept in mind throughout the proof.

1. If $v_2(l) \leq \mathrm{c}_2 + 2$, we make use of the fact that $\gcd(m, 2 + m'\mu) = 1$. Hence, by Theorem 1.3.2, there exists an autormorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that

$$\sigma_{|A[m]} = [(2 + m'\mu)^{\mathrm{c}}]_{|A[m]}.$$

This Galois automorphism maps Q to

$$Q^\sigma = [2^{\mathrm{c}}]Q + \sum_{1\leq\gamma\leq \mathrm{c}} \left[\binom{\mathrm{c}}{\gamma} 2^{\mathrm{c}-\gamma}(m'\mu)^\gamma\right]Q \tag{1.3.8}$$

Firstly, since $2 \nmid m'\mu$, we have that $[(m'\mu)^\gamma]Q$ is a point of order $2^{v_2(l)}$, for every $\gamma = 1, \ldots, \mathrm{c}$. In particular, we have $[\binom{\mathrm{c}}{\gamma}2^{\mathrm{c}-\gamma}(m'\mu)^\gamma]Q \in A[2^{\mathrm{c}_2+2}]$. We derive from this that for some point $P \in A[2^{\mathrm{c}_2+2}] \subset A[4\mathrm{c}]$ we have

$$Q^\sigma = [2^{\mathrm{c}}]Q - P.$$

On the other hand, by definition of M in (1.3.3), $v_2(M) \geq \mathrm{c}_2+2 \geq v_2(l)$. Therefore, $v_2(m) = v_2(M)$. In addition, using the congruence in (1.3.7), we obtain

$$\sigma_{|A[M]} = [(2 + 2^{-v_2(M)}M)^{\mathrm{c}}]_{|A[M]}.$$

2. Assume next $\mathrm{c}_2 + 3 \leq v_2(l) \leq \mathrm{c}_2 + \beta + 1 + v_2(M) = \mathrm{c}_2 + v_2(N)$, with $N = 2^{\beta+1}M$ as in (1.3.5).

   Since $2 \leq v_2(N) - 1 < v_2(M)$, we have that $1 + 2^{v_2(M)-1}m'\mu$ is an integer coprime to m. Hence, by Theorem 1.3.2, there exists an automorphism $\rho \in \mathrm{Gal}(\overline{K}/K)$ such that
$$\rho_{|A[m]} = [(1 + 2^{v_2(N)-1}m'\mu)^{\mathrm{c}}]_{|A[m]}.$$
   This Galois automorphism maps Q to

$$Q^\rho = Q + [\mathrm{c}2^{v_2(N)-1}m'\mu]Q + \sum_{2\leq\gamma\leq\mathrm{c}} \left[\binom{\mathrm{c}}{\gamma}(2^{v_2(N)-1}m'\mu)^\gamma\right]Q. \tag{1.3.9}$$

   Let $\gamma \geq 2$. Since $v_2(2^{v_2(N)-1}m'\mu) = v_2(N) - 1 \geq 2$, Lemma 1.3.5 gives

$$v_2\left(\binom{\mathrm{c}}{\gamma}(2^{v_2(N)-1}m'\mu)^\gamma\right) \geq v_2(N) - 1 + \mathrm{c}_2 + 1 = v_2(N) + \mathrm{c}_2 \geq v_2(l).$$

   Thus, for the corresponding terms in (1.3.9), we have that $[\binom{\mathrm{c}}{\gamma}(2^{v_2(N)-1}m'\mu)^\gamma]Q = 0$. Moreover, since $v_2(\mathrm{c}2^{v_2(N)-1}m'\mu) = \mathrm{c}_2 + v_2(N) - 1 \geq v_2(l) - 1$, we have that $[\mathrm{c}2^{v_2(N)-1}m'\mu]Q$ is a point of order dividing 2. From this we derive that for some point $P \in A[2]$, we have
$$Q^\rho = Q - P.$$

   Using the congruence in (1.3.7), ρ acts on $A[M]$ as

$$\rho_{|A[M]} = [(1 + 2^{v_2(N)-1-v_2(M)}M)^{\mathrm{c}}]_{|A[M]} = [(1 + 2^\beta M)^{\mathrm{c}}]_{|A[M]}.$$

3. For the last case, assume $v_2(l) \geq \max\{\mathrm{c}_2 + 3, \mathrm{c}_2 + v_2(N) + 1\}$.

   Since $1 < v_2(l) - \mathrm{c}_2 - 1 \leq v_2(m)$, we have that $1 + 2^{v_2(l)-\mathrm{c}_2-1} m'\mu$ is an integer coprime to m. Hence, by Theorem 1.3.2, there exists an automorphism $\tau \in \mathrm{Gal}(\overline{K}/K)$ such that
$$\tau_{|A[m]} = [(1 + 2^{v_2(l)-\mathrm{c}_2-1} m'\mu)^{c}]_{|A[m]}.$$

   This Galois automorphism maps Q to
$$Q^{\tau} = Q + [\mathrm{c}2^{v_2(l)-\mathrm{c}_2-1} m'\mu]Q + \sum_{2 \leq \gamma \leq \mathrm{c}} \Big[\binom{\mathrm{c}}{\gamma}(2^{v_2(l)-\mathrm{c}_2-1} m'\mu)^{\gamma}\Big]Q \qquad (1.3.10)$$

   Similarly to the preceding case, $v_2(2^{v_2(l)-\mathrm{c}_2-1} m'\mu) \geq 2$ and Lemma 1.3.5 yields the equality $[\binom{\mathrm{c}}{\gamma}(2^{v_2(l)-\mathrm{c}_2-1} m'\mu)^{\gamma}]Q = 0$, for $\gamma = 2, \ldots, \mathrm{c}$. Moreover, since $v_2(\mathrm{c}2^{v_2(l)-\mathrm{c}_2-1} m'\mu) = v_2(l) - 1$, we have that $[\mathrm{c}2^{v_2(l)-\mathrm{c}_2-1} m'\mu]Q$ is a point of order 2. From this we derive that for some point $P \in A[2] \setminus \{0\}$ we have
$$Q^{\tau} = Q - P.$$

   On the other hand, by the congruence in (1.3.7)
$$\tau_{|A[M]} = [(1 + 2^{v_2(l)-\mathrm{c}_2-1-v_2(M)} M)^{\mathrm{c}}]_{|A[M]}.$$

   Furthermore, notice that $\alpha := v_2(l) - \mathrm{c}_2 - 1 - v_2(M) \geq v_2(N) - v_2(M) = \beta + 1$. This implies that $\alpha \in \mathcal{N}(0)$, and so there exists a Galois automorphism $\tau' \in \mathrm{Gal}(\overline{K}/K)$ that is not necessarily τ, but coincides with it on $A[M]$, such that $V^{\tau'} = V$. Since V is defined over $K(A[M])$, $V^{\tau} = V^{\tau'} = V$.

□

By means of the closed immersion fixed at the beginning of this section, we may identify every subvariety $X \subset A$ with its image by ι. This allows us to consider the degree of X as the degree of the Zariski closure of $\iota(X)$ in $\mathbb{P}^n$. The definition of this degree depends on the chosen immersion.

It is essential to have some control over the degree with the operations in A. First, it is invariant by translations in A, see for instance [42, Lemme 7]. For the multiplication map by $k \in \mathbb{N}^*$, a result of Hindry [42, Lemme 6(ii)] gives, for every subvariety $X \subset A$,
$$\deg([k]^{-1}(X)) = k^{2\,\mathrm{codim}_A(X)} \deg(X). \qquad (1.3.11)$$

We have all the ingredients to prove the following result.

Proposition 1.3.8. *Let $V \subset A$ be an irreducible subvariety defined over $K(A_{tors})$, and $\varphi : A \twoheadrightarrow B$ be a homomorphism of algebraic groups defined over K as in* (1.3.2)*. Assume that the β in* (1.3.4) *relative to $\varphi(V) \subset B$ attains its minimum at 0. Then, there exist two Galois automorphisms $\sigma, \rho \in \mathrm{Gal}(\overline{K}/K)$, such that $\overline{V_{tors}}$ is contained in*

$$V' = \bigcup_{P \in B[4\mathrm{c}]} [2^{\mathrm{c}}]^{-1}(V^{\sigma} + \varphi^{-1}(P)) \;\cup \bigcup_{P \in B[2]} (V^{\rho} + \varphi^{-1}(P)) \;\cup \bigcup_{P \in B[2] \setminus \{0\}} (V + \varphi^{-1}(P)),$$

and $V' \cap V \subsetneq V$.

Proof. Since φ is a group homomorphism, $\varphi(V \cap A_{\mathrm{tors}}) = \varphi(V) \cap B_{\mathrm{tors}}$. In fact, since $\mathrm{Ker}(\varphi) = \mathrm{Stab}(V)$, we have $\varphi^{-1}(\overline{\varphi(V)_{\mathrm{tors}}}) = \overline{V_{\mathrm{tors}}}$. Notice that the variety V' is the preimage of the variety we obtain by applying Proposition 1.3.7 to $\varphi(V)$. This already gives the inclusion $\overline{V_{\mathrm{tors}}} \subset V'$. Then, to prove $V' \cap V \subsetneq V$ it is enough to proof that $\varphi(V) \cap \varphi(V') \subsetneq \varphi(V)$. To simplify the notations, let us assume that V has trivial stabilizer, so $\varphi = \mathrm{Id}$ and $B = A$ in the rest of the proof.

Take the Galois automorphisms $\sigma, \rho \in \mathrm{Gal}(\overline{K}/K)$ to be as in Proposition 1.3.7. We separate the proof in three cases, corresponding to each group of varieties in the expression of V'.

1. First, we show that $V \not\subset [2^{\mathrm{c}}]^{-1}(V^{\sigma} + P)$, for every $P \in A[4\mathrm{c}]$.

   Assume that there is a such point $P \in A[4\mathrm{c}]$ such that $V \subset [2^{\mathrm{c}}]^{-1}(V^{\sigma} + P)$. Then,

   $$\bigcup_{R \in A[2^{\mathrm{c}}]} V + R \subset [2^{\mathrm{c}}]^{-1}(V^{\sigma} + P).$$

   On the left-hand side we have $(2^{\mathrm{c}})^{2g}$ different varieties of degree $\deg(V)$, because V is assumed to have trivial stabilizer. This gives a variety of degree $(2^{\mathrm{c}})^{2g} \deg(V)$. However, due to (1.3.11), we have that the variety on the right is of degree

   $$(2^{\mathrm{c}})^{2\,\mathrm{codim}_A(V)} \deg(V) < (2^{\mathrm{c}})^{2g} \deg(V).$$

   This yields a contradiction.

2. Next, we show that $V \not\subset V^{\rho} + P$, for every $P \in A[2]$. Let M and β be as defined in (1.3.3) and (1.3.4), respectively. Recall that this case only arises whenever $\beta + v_2(M) \geq 2$. As mentioned above, we are under the hypothesis that $\beta = \beta(0)$.

   Assume that $V^{\rho} + P = V$ for some $P \in A[2]$. Let $R \in A[2^{\mathrm{c}_2+\beta+1}M]$ such that $[\mathrm{c}\,2^{\beta}M]R = [2^{\mathrm{c}_2+\beta}M]R = P$. By Lemma 1.3.6, $\beta < -\mathrm{c}_2$, and we have $R \in A[M]$. Then, by the explicit expression of the action of ρ on $A[M]$, we get

   $$R^{\rho} = [(1 + 2^{\beta}M)^{\mathrm{c}}]R = R + P + \sum_{2 \leq \gamma \leq \mathrm{c}} \Big[\binom{\mathrm{c}}{\gamma}(2^{\beta}M)^{\gamma}\Big]R. \tag{1.3.12}$$

Let $\gamma \geq 2$. Since $v_2(2^\beta M) \geq 2$, Lemma 1.3.5 gives

$$v_2\left(\binom{\mathrm{c}}{\gamma}(2^\beta M)^\gamma\right) \geq \beta + v_2(M) + \mathrm{c}_2 + 1.$$

Thus, for the corresponding summands in (1.3.12), we have that $2^{\mathrm{c}_2+\beta+1}M$ divides $\binom{\mathrm{c}}{\gamma}(2^\beta M)^\gamma$. Since $R \in A[2^{\mathrm{c}_2+\beta+1}M]$, this gives $[\binom{\mathrm{c}}{\gamma}](2^\beta M)^\gamma]R = 0$. Then

$$(V+R)^\rho = V^\rho + R^\rho = V - P + (R+P) = V + R.$$

This implies that $V + R$ is fixed by ρ, and we conclude that $\beta \in \mathcal{N}(R)$.

Take $\alpha \geq \beta + 1 = \beta(0) + 1$. By definition of $\beta(0)$, there exists an automorphism $\rho_0 \in \mathrm{Gal}(\overline{K}/K)$ such that $\rho_{0|A[M]} = [(1 + 2^\alpha M)^{\mathrm{c}}]$ and $\rho_0(V) = V$. Notice that $2^{\beta+1}M$ divides $2^\alpha M$. So by expanding R^{ρ_0} as in (1.3.12), we readily obtain $R^{\rho_0} = R$. Hence, we also have $(V+R)^\rho = V^{\rho_0} + R^{\rho_0} = V + R$, which shows that $\alpha \in \mathcal{N}(R)$.

From the above statements, we conclude that $\beta + \mathbb{N}_{\geq 0} \subset \mathcal{N}(R)$. Therefore,

$$\beta(R) < \beta,$$

and this contradicts the minimality of β.

3. Finally, the fact that $V \neq V + P$ for every $P \in A[2] \setminus \{0\}$ follows directly from the assumption that V has trivial stabilizer.

□

1.3.2 The case of a curve

Proposition 1.3.8 is enough to give an explicit upper bound (modulo the constant c) for the number of torsion points in a curve of genus greater than 1 in an Abelian variety. Indeed, the result we give in this section is the analogue of Beukers and Smyth's one in [5]; that is an explicit upper bound for the number of torsion points in V, when V is a curve.

Proposition 1.3.9. *Let $C \subset A$ be an irreducible algebraic curve of genus greater than* 1*. Then*

$$\# C_{\mathrm{tors}} \leq (2^{2g\mathrm{c}+4g-2\mathrm{c}}\, \mathrm{c}^{2g} + 2^{2g+1} - 1)\deg(C)^2.$$

Proof. The result follows directly from computing the degree of V' in Proposition 1.3.7, which is $((4\mathrm{c})^{2g}\,(2^{\mathrm{c}})^{2(g-1)} + 2^{2g} + 2^{2g} - 1)\deg(C)$. So, due to Proposition 1.3.8 a straightforward application of Bézout's theorem yields the result. □

Remark. *A mild improvement can be made to this bound if we assume the Abelian variety is in fact the Jacobian J of a smooth irreducible projective curve C, with the closed immersion $J \hookrightarrow \mathbb{P}^n$, given by taking m times the theta-divisor coming from the Abel maps, choosing m so that the resulting divisor is very ample. So C is of genus $g > 1$, and by Poincaré's formula (see for instance [60, Equation (4)]),* $\deg(C) = m\,g$*. In this case C and J are defined over the same number field. This implies that in the proof of Proposition 1.3.7 the second case does not occur. Therefore, we do not need to consider the irreducible components $V^\rho + P$ of V', and we have that*

$$\# C_{\mathrm{tors}} \leq m(2^{2g\mathrm{c}+4g-2\mathrm{c}}\,\mathrm{c}^{2g} + 2^{2g} - 1)g^2.$$

1.3.3 Degrees of definition and Hilbert functions

In the case of treating varieties of dimension > 1, an iterative use of Bézout's theorem would give a bound which is doubly exponential in the degree of the variety, as was the case in [1]. It is therefore helpful to introduce the equivalent notions of degree we used in the toric case.

We briefly recall the definitions. By identifying every subvariety of A with its image in $\mathbb{P}^n$, we say that the degree of (complete) definition of $V \subset A$ is the minimal degree $\delta(V)$ such that V is the intersection of a family of hypersurfaces in $\mathbb{P}^n$ of degree at most $\delta(V)$. On the other hand, the degree of incomplete definition of V is the minimal degree $\delta_0(V)$ such that the irreducible components of V are also irreducible components of the intersection of a family of hypersurfaces in $\mathbb{P}^n$ of degree at most $\delta_0(V)$.

Contrary to the toric case, the degree of complete (respectively incomplete) definition does not necessarily behave as the usual degree with respect to translations in A. First we present the following consequence to a result of Lange and Ruppert [51, Theorem].

Lemma 1.3.10. *The translations in A can be defined locally in terms of homogeneous polynomials in $\overline{\mathbb{K}}[X_1, \ldots, X_n]$ of degree at most* 2*.*

As a consequence, we deduce the following lemma.

Lemma 1.3.11. *Let V be a subvariety of A.*

(i) For any point $P \in A$, we have

$$\delta(P + V) \leq 2\,\delta(V) \quad \text{and} \quad \delta_0(P + V) \leq 2\,\delta_0(V).$$

(ii) Assume $K_V \subset K(A_{tors})$. For any finite subset $T \subset A_{tors} \times \mathrm{Gal}(\overline{K}/K)$ of cardinality $\#T = t$, we have

$$\delta_0\Big(\bigcup_{(P,\phi)\in T} P + V^\phi\Big) \leq 4\,t\,\delta_0(V).$$

Proof. Assertion (i) is a direct consequence of Lemma 1.3.10.

To prove (ii), let $n \in \mathbb{N}_{>0}$ be such that $K_V \subset K(A[n])$, and $[n]P = 0$ for every P appearing as the first coordinate of a pair in T. We may then replace the T in the statement by

$$\{(P, \phi_{|K(A[n])}) \mid (P, \phi) \in T\} \subset A[n] \times \mathrm{Gal}(K(A[n])/K).$$

Also notice that, for each $P_1, P_2 \in A[n]$ and each $\phi_1, \phi_2 \in \mathrm{Gal}(K(A[n])/K)$, we have

$$P_2 + (P_1 + V^{\phi_1})^{\phi_2} = P_2 + \phi_2^{-1}(P_1) + V^{\phi_1\,\phi_2}.$$

This relation defines a natural structure of semidirect product on $A[n] \rtimes \mathrm{Gal}(K(A[n])/K)$, given by

$$(P_1, \phi_1) \cdot (P_2, \phi_2) = (P_2 + \phi_2^{-1}(P_1), \phi_1\phi_2);$$

where the inverse of an element (P, ϕ) is $(\phi(-P), \phi^{-1})$.

By the definition of degree of incomplete definition, there exists a subvariety $X \subset A$ such that V is an irreducible component of X and $\delta_0(V) = \delta(X)$. We denote by G the group

$$G = \langle a \cdot b^{-1} \mid a, b \in T\rangle \subset A[n] \rtimes \mathrm{Gal}(K(A[n])/K),$$

and by S the subset of G consisting of the pairs $(P, \phi) \in G$ such that $P + V^{\phi}$ is imbedded in X. Consider then the variety

$$\widetilde{X} = X \cap \Big(\bigcap_{(P,\phi)\in S} \phi(-P) + X^{\phi^{-1}} \Big).$$

By construction, V is an irreducible component of $\widetilde{X}$, and from (i) we have $\delta(\widetilde{X}) \leq 2\delta(X) = 2\delta_0(V)$. Moreover, the following claim holds.

Claim. There is no $(P, \phi) \in G$ for which $P + V^{\phi}$ is imbedded in $\widetilde{X}$.

Proof of the claim. Assume that $P + V^{\phi}$ is properly included in $\widetilde{X}$, for some $(P, \phi) \in G$. Since $\widetilde{X} \subset X$, $P + V^{\phi}$ is also properly included in X, which means $(P, \phi) \in S$. By induction, this yields $(P_k, \phi^k) = (P, \phi)^k \in S$ for all k. Assume $(P_k, \phi^k) \in S$, then $\widetilde{X} \subset \phi^k(-P_k) + X^{\phi^{-k}}$ and so $P + V^{\phi}$ is properly included in $\phi^k(-P_k) + X^{\phi^{-k}}$, which implies $(P_{k+1}, \phi^{k+1}) \in S$. Hence, taking $k = \mathrm{ord}\big((P, \phi)\big)$, we have $(0, \mathrm{Id}) \in S$ that contradicts the fact that V is an irreducible component of X. $\square$

Let us consider the subvariety

$$Y = \bigcup_{(P,\phi)\in T} P + \widetilde{X}^{\phi}.$$

Then $P + V^{\phi} \subset Y$, for every $(P, \phi) \in T$. Let us assume that there is a pair $(P, \phi) \in T$ such that $P + V^{\phi}$ is properly included in Y. This means that there is a $(Q, \psi) \in T$ such that $P + V^{\phi}$ is properly included in $Q + \widetilde{X}^{\psi}$. Thus

$$\psi(-Q) + (P + V^{\phi})^{\psi^{-1}} = \psi(-Q + P) + V^{\phi\,\psi^{-1}}$$

is properly included in $\widetilde{X}$. This contradicts the claim, since

$$(P,\phi)\cdot(Q,\psi)^{-1}=(\psi(-Q+P),\phi\,\psi^{-1})\in G.$$

So $P+V^{\phi}$ is an irreducible component of Y, for every $(P,\phi)\in T$. Moreover, notice that

$$\delta(\bigcup_{i=1}^{t}W_i)\leq\sum_{i=1}^{t}\delta(W_i),$$

for every family of varieties $W_1,\ldots,W_t\subset\mathbb{P}^n$. Hence, by (i), we also have $\delta(Y)\leq\sum\delta(2t\delta(\widetilde{X})$. Assertion (ii) follows then from the fact that $\delta(\widetilde{X})\leq 2\delta_0(V)$. □

Next, let us recall that if the closure of V in $\mathbb{P}^n$ is defined by the homogeneous radical ideal I in $\overline{\mathbb{Q}}[\boldsymbol{x}]$; for $\nu\in\mathbb{N}$, $H(V;\nu)=\dim(\overline{\mathbb{Q}}[\boldsymbol{x}]/I)_\nu$ denotes the Hilbert function. And we also recall the upper and lower bounds on the Hilbert functions due to Chardin [23], and Chardin and Philippon [24], respectively. Let $X\subseteq\mathbb{P}^n$ be an equidimensional variety of dimension d. Then, for every $\nu\in\mathbb{N}$,

$$H(X;\nu)\leq\binom{\nu+d}{d}\deg(X). \tag{1.3.13}$$

Moreover, if $\nu>m=\mathrm{codim}_{\mathbb{P}^n}(X)(\delta_0(X)-1)$,

$$H(X;\nu)\geq\binom{\nu+d-m}{d}\deg(X). \tag{1.3.14}$$

By means of these bounds for the Hilbert function, we obtain the following result:

Lemma 1.3.12. *Let V be an irreducible proper subvariety of A of dimension $d>0$, such that $K_V\subset K(A_{tors})$ and $V\neq\overline{V_{tors}}$. Let $\phi\in\mathrm{Gal}(\overline{K}/K)$, $P\in A_{tors}$ and $k\geq 2$ be an integer.*

(i) If $P+V^{\phi}\neq V$, then there exists a hypersurface Z of $\mathbb{P}^n$ of degree at most $8(2d+1)\,\mathrm{codim}_{\mathbb{P}^n}(V)\delta_0(V)$ such that $P+V^{\phi}\subset Z$ and $V\cap Z\subsetneq V$.

(ii) If $V\not\subset[k]^{-1}(P+V^{\phi})$, then there exists a hypersurface Z' of $\mathbb{P}^n$ of degree at most $8k^{2g}(2d+1)\,\mathrm{codim}_{\mathbb{P}^n}(V)\delta_0(V)$ such that $[k]^{-1}(P+V^{\phi})\subset Z'$ and $V\cap Z'\subsetneq V$.

Proof. We start by proving (i). Notice that $P+V^{\phi}$ is an irreducible subvariety of A. By (1.3.13), for any $\nu\in\mathbb{N}$,

$$H(P+V^{\phi};\nu)\leq\binom{\nu+d}{d}\deg(V).$$

Denote $\widetilde{V} = V \cup (P + V^{\phi})$. This is an equidimensional variety (of dimension d) of degree $2\deg(V)$. Using (1.3.14), for any $\nu > m$,

$$H(\widetilde{V};\nu) \geq \binom{\nu+d-m}{d} 2\deg(V),$$

where $m = \operatorname{codim}_{\mathbb{P}^n}(\widetilde{V})(\delta_0(\widetilde{V}) - 1)$. Fix $\nu = m(2d+1)$. We obtain the following inequality:

$$\begin{aligned}\frac{H(P+V^{\phi};\nu)}{H(\widetilde{V};\nu)} &\leq \frac{1}{2}\binom{\nu+d}{d}\binom{\nu+d-m}{d}^{-1} \\ &\leq \frac{1}{2}\left(1+\frac{m}{\nu-m}\right)^{d} = \frac{1}{2}\left(1+\frac{1}{2d}\right)^{d} \leq \frac{1}{2}\mathrm{e}^{1/2} < 1.\end{aligned}$$

Thereby, there is a hypersurface Z of $\mathbb{P}^n$ of degree ν such that $P+V^{\phi} \subset Z$ and $\widetilde{V} \cap Z \subsetneq \widetilde{V}$. In particular, $V \not\subset Z$. Moreover, by Lemma 1.3.11(ii) we have $\delta_0(\widetilde{V}) \leq 8\delta_0(V)$. Then, we obtain the bound on the degree of Z:

$$\deg(Z) \leq 8(2d+1)\operatorname{codim}_{\mathbb{P}^n}(V)\delta(V),$$

concluding the proof of (i).

We now turn to prove assertion (ii). For simplicity, we denote $W = [k]^{-1}(P + V^{\phi})$. It is an equidimensional subvariety of A of dimension d. As a consequence to (1.3.11), we have $\deg(W) = k^{2\operatorname{codim}_A(V)}\deg(V)$. By (1.3.13), for any $\nu \in \mathbb{N}$,

$$H(W;v) \leq \binom{\nu+d}{d} k^{2\operatorname{codim}_A(V)}\deg(V).$$

Denote $\widetilde{W} = \bigcup_{Q\in[k]^{-1}\operatorname{Stab}(V)}(Q+V)$. Let $\varphi : A \to B$ be the isogeny that trivializes the stabilizer as in (1.3.2), and $r = \operatorname{codim}_A(\operatorname{Stab}(V)) = \dim(B)$. Since $[k]\circ\varphi = \varphi\circ[k]$ we have that $\widetilde{W}$ is an equidimensional subvariety of A of dimension d and degree $k^{2r}\deg(V)$. Using (1.3.14), for any $\nu > m$,

$$H(\widetilde{W};\nu) \geq \binom{\nu+d-m}{d} k^{2r}\deg(V),$$

where $m = \operatorname{codim}_{\mathbb{P}^n}(\widetilde{W})(\delta_0(\widetilde{W}) - 1)$. Notice that from the fact that $V \neq \overline{V_{\mathrm{tors}}}$ and $r > 0$, since $V \neq A$, we have $\operatorname{codim}_A(V) < r$. So $k^{2\operatorname{codim}_A(V)-2r} \leq k^{-2} < \mathrm{e}^{-1}$. Fix $\nu = m(2d+1)$. We obtain the following inequality:

$$\frac{H(W;\nu)}{H(\widetilde{W};\nu)} \leq k^{2\operatorname{codim}_A(V)-2r}\binom{\nu+d}{d}\binom{\nu+d-m}{d}^{-1} \leq k^{2\operatorname{codim}_A(V)-2r}\mathrm{e}^{1/2} < 1.$$

Thereby, there is a hypersurface Z_0' of $\mathbb{P}^n$ of degree ν such that $W \subset Z_0'$ and $\widetilde{W} \cap Z_0' \subsetneq \widetilde{W}$. In particular, there is a $Q_0 \in [k]^{-1}\operatorname{Stab}(V)$ such that $Z_0' \cap (Q_0 + V) \subsetneq Q_0 + V$. Notice that $Z_0' \cap A$ is a hypersurface in A since it intersects properly $Q_0 + V \subset A$.

Let $X = -Q_0 + (Z_0' \cap A)$, then $V \cap X \neq V$. On the other hand, for every $Q \in [k]^{-1}\operatorname{Stab}(V)$, we have $Q + [k]^{-1}(P + V^{\phi}) = [k]^{-1}(P + V^{\phi})$. This implies $W \subset X$. By Lemma 1.3.10, there is a hypersurface Z' of degree $2\deg(Z_0') = 2\nu$ such that $X = Z' \cap A$. This hypersurface satisfies $W \subset Z'$ and $V \cap Z' \subsetneq V$. Moreover, by Lemma 1.3.11(ii) we have $\delta_0(\widetilde{W}) \leq 4k^{2r}\delta_0(V) \leq 4k^{2g}\delta_0(V)$. Then, we obtain the bound on the degree of Z':

$$\deg(Z') \leq 8\, k^{2g}(2d+1)\operatorname{codim}_{\mathbb{P}^n}(V)\delta_0(V),$$

which ends the proof of (ii). □

1.3.4 Interpolation and proof of the theorem

We start by presenting the key element for the proof of the main theorem.

Proposition 1.3.13. *Let $V \subset A$ be an irreducible variety of dimension $d > 0$, such that $V \neq \overline{V_{tors}}$. Then there exists a hypersurface $Z \subset \mathbb{P}^n$ of degree at most*

$$(2^{2gc+4g+5}\,c^{2g} + 2^{2g+6})(2d+1)(n-d)\,\delta_0(V), \tag{1.3.15}$$

such that $\overline{V_{tors}} \subset V \cap Z \subsetneq V$.

Proof. First assume $K_V \subset K(A_{\mathrm{tors}})$. Moreover, let us assume that β in (1.3.4) for $\varphi(V)$ attains its minimum at 0. We apply Lemma 1.3.12 to the distinct components in that appear in the union defining V' in Proposition 1.3.8. Let us consider the notations as in Proposition 1.3.8. For each $P \in B[4c]$, we have that $V \not\subset [2]^{-1}(V^{\sigma} + \varphi^{-1}(P))$, which gives a hypersurface $Z_{\sigma,P}$ of degree bounded as in Lemma 1.3.12(i) of the lemma. Moreover, for each $P \in B[2]$ we have that $V \neq V^{\rho} + \varphi^{-1}(P)$ and $V \neq V + \varphi^{-1}(P)$, which gives respectively a hypersurface $Z_{\rho,P}$ and Z_P as in Lemma 1.3.12(ii). Then, for $Z = (\bigcup_{P\in B[4c]} Z_{\sigma,P}) \cup (\bigcup_{P\in B[2]} Z_{\rho,P}) \cup (\bigcup_{P\in B[2]} Z_P)$, we have that $V \cap V' \subset V \cap Z$ and $V \not\subset Z$. Moreover, the degree of Z is at most

$$\begin{aligned}\sum_{P\in B[4c]} 8\cdot(2^c)^{2g}(2d+1)(n-d)\delta_0(V) + 2\sum_{P\in B[2]} 8(2d+1)(n-d)\delta_0(V)\\ \leq (2^{2gc+4g+3}c^{2g} + 2^{2g+4})(2d+1)(n-d)\delta_0(V).\end{aligned} \tag{1.3.16}$$

Notice that the inequality comes from the implicit use of $\dim(B) \leq \dim(A)$.

If β in (1.3.4) for $\varphi(V)$ does not attain its minimum at 0, let $R \in B[M] \setminus \{0\}$ be such that $\beta = \beta(R)$. Fix an element $R' \in \varphi^{-1}(R)$. Then, since $\delta_0(V + R') \leq 2\delta_0(V)$ by Lemma 1.3.11(ii), there exists a hypersurface of degree at most 2 times the expression in (1.3.16), such that $\overline{(V+R')_{\mathrm{tors}}} \subset (V+R') \cap Z' \subsetneq V+R'$. Then $Z = Z' - R'$ is a hypersurface in $\mathbb{P}^n$ such that

$$\overline{V_{\mathrm{tors}}} = \overline{(V+R')_{\mathrm{tors}}} - R' \subset V \cap Z \subsetneq V.$$

In addition, the degree of Z is $2\deg(Z')$ by Lemma 1.3.10, which is bounded above by the expression in (1.3.15).

If $K_V \not\subset K(A_{\mathrm{tors}})$, as consequence to Proposition 1.3.4, for every non-trivial Galois automorphism $\varsigma \in \mathrm{Gal}(K_V/(K_V \cap K(A_{\mathrm{tors}})))$, one has $\overline{V_{\mathrm{tors}}} \subset V \cap V^{\varsigma} \subsetneq V$. First, one uses the fact that $\delta_0(V \cup V^{\varsigma}) \leq 2\delta_0(V)$ to prove Lemma 1.3.12(i) for V and V^{ς}, with $P = 0$. This concludes the proof, since $8(2d+1)(n-d)\delta_0(V)$ is at most the value in (1.3.15). $\square$

The main theorem of this section states the following.

Theorem 1.3.14. *Let $V \subset A$ be a subvariety of dimension $d > 0$. For $j = 0, \ldots, d$, let V^j_{tors} denote the j-equidimensional part of $\overline{V_{tors}}$. Then, for every $j = 0, \ldots, d$,*

$$\deg(V^j_{tors}) \leq c_j \; \delta(V)^{g-j}.$$

where

$$c_j = ((2^{2g\mathrm{c}+4g+5}\mathrm{c}^{2g} + 2^{2g+6})(2g-1)(n-1))^{(g-j)d} \deg(A).$$

Proof. For $j = 0, \ldots, d$, let us denote by X^j the j-equidimensional part of V. We also fix

$$\theta = ((2^{2g\mathrm{c}+4g+5}\mathrm{c}^{2g} + 2^{2g+6})(2g-1)(n-1))^{d} \; \delta(V).$$

We first apply the result of Philippon [66, Corollaire 5] as follows. With the notation used by Philippon in *loc. cit.*, we take $m = g$, $S = A$, $\varphi = \iota : A \hookrightarrow \mathbb{P}^n$, $\delta = \theta$, and $Z_1, \ldots, Z_l$ the hypersurfaces in $\mathbb{P}^n$ of degree at most $\delta(V)$ such that $V = Z_1 \cap \cdots \cap Z_l$. (after identifying V and A with their image in $\mathbb{P}^n$). In particular, $Z_1 \cap \cdots \cap Z_l = A \cap Z_1 \cap \cdots \cap Z_l$. Then, from the result of Philippon applied to the cycle $S_l = A \cdot Z_1 \cdots Z_l$, we deduce

$$\sum_{j=0}^{d} \theta^j \deg(X^j) \leq \theta^n \; \cdot \deg(A). \tag{1.3.17}$$

Then, following straightforwardly the double induction in the proofs of Theorems 1.2.17 and 1.2.18, with Proposition 1.3.13 at the place of Proposition 1.2.16; one obtains the inequality

$$\sum_{j=0}^{d} \theta^j \deg(V^j_{\mathrm{tors}}) \leq \sum_{j=0}^{d} \theta^j \deg(X^j). \tag{1.3.18}$$

The upper bound in the theorem then follows from combining (1.3.17) and (1.3.18). $\square$

Remark. *As a final remark we should precise that the upper bound given by Theorem 1.3.14 is effective up to the constant* c.

Chapter 2

An arithmetic Bernštein-Kušnirenko theorem

In this chapter we present the results included in the joint work [59]. We study the height(s) of zero-cycles defined by a system of Laurent polynomials, in terms of mixed integrals of specific concave functions. In doing so we provide an arithmetic analogue of Bernštein-Kušnirenkos upper bound on the number of solutions of a such system.

2.1 Introduction

The classical Bernštein-Kušnirenko theorem bounds the number of isolated zeros of a system of Laurent polynomials over a field, in terms of the mixed volume of their Newton polytopes. This result, initiated by Kušnirenko and put into final form by Bernštein, is also known as the BKK theorem to acknowledge Khovanskiĭ's contributions to this subject. It shows how a geometric problem (the counting of the number of solutions of a system of equations) can be translated into a combinatorial, simpler one. It is commonly used to predict when a given system of equations has a small number of solutions. As such, it is a cornerstone of polynomial equation solving and has motivated a large amount of work and results over the past 25 years, see for instance [36, 67, 83] and the references therein.

Let $\mathbb{K}$ be a field, and fix an algebraic closure $\overline{\mathbb{K}}$. Let $M \simeq \mathbb{Z}^n$ be a lattice, and set

$$\mathbb{K}[M] = \bigoplus_{m\in M} \mathbb{K}\cdot\chi^m \simeq \mathbb{K}[x_1^{\pm1},\dots,x_n^{\pm1}]$$

for its group $\mathbb{K}$-algebra, and

$$\mathbb{T}_M = \operatorname{Spec}(\mathbb{K}[M]) \simeq \mathbb{G}_{\mathrm{m},\mathbb{K}}^n$$

for its algebraic torus over $\mathbb{K}$. For a family of Laurent polynomials $f_1, \dots, f_n \in \mathbb{K}[M]$, we denote by $Z(f_1, \dots, f_n)$ the 0-cycle of $\mathbb{T}_M$ given by the isolated solutions of the system of equations

$$f_1 = \cdots = f_n = 0$$

with their corresponding multiplicities (Definition 2.2.8).

Set $M_{\mathbb{R}} = M \otimes \mathbb{R} \simeq \mathbb{R}^n$. Let vol_M be the Haar measure on $M_{\mathbb{R}}$ normalized so that M has covolume 1, and let MV_M be the corresponding mixed volume function (Definiton 2.2.7). For $i = 1, \dots, n$, let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of f_i. The BKK theorem [4, 47] amounts to the upper bound

$$\deg(Z(f_1, \dots, f_n)) \leq \mathrm{MV}_M(\Delta_1, \dots, \Delta_n), \tag{2.1.1}$$

which is an equality when the f_i's are generic with respect to their Newton polytopes, see also Theorem 2.2.10.

When dealing with Laurent polynomials over a field with an arithmetic structure like the field of rational numbers, it is also important to control the arithmetic complexity or *height* of their zero set. In this chapter, we present an arithmetic version of the BKK theorem, bounding the height of the isolated zeros of a system of Laurent polynomials over such a field. It is a refinement of the arithmetic Bézout theorem that takes into account the finer monomial structure of the system.

Suppose that $\mathbb{K}$ is endowed with a set of places $\mathfrak{M}$, so that the pair $(\mathbb{K}, \mathfrak{M})$ is an adelic field (Definition 2.3.1). Each place $v \in \mathfrak{M}$ corresponds to an absolute value $|\cdot|_v$ on $\mathbb{K}$ and a weight $n_v > 0$. We assume that this set of places satisfies the *product formula*, namely, for all $\alpha \in \mathbb{K}^{\times}$,

$$\sum_{v \in \mathfrak{M}} n_v \log |\alpha|_v = 0.$$

The classical examples of adelic fields satisfying the product formula are number fields and finite extensions of function fields of curves. These are called global fields in [19], and are more general than the usual notion of global fields, since neither the base field of the function fields is required to be finite nor the extension is assumed separable.

Let X be toric compactification of $\mathbb{T}_M$ and $\overline{D}_0$ a nef toric metrized divisor on X as in Definition 2.3.25. This data gives a notion of height for 0-cycles of X (Definitions 2.3.10 and 2.3.14). Then, for a family of Laurent polynomials $f_1, \dots, f_n \in \mathbb{K}[M]$, the height

$$\mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n))$$

is a nonnegative real number. It is our aim to bound this quantity in terms of the monomial expansion of the f_i's.

The first arithmetic analogue of the BKK theorem was proposed by Maillot [56, Corollaire 8.2.3], who considered the case of canonical toric metrics. His result is not completely effective, as explained in [82, Remarque 4.2]. Another result in this direction was obtained by Sombra for the unmixed case and also canonical toric metrics [82, Théoreme 0.3]. In this chapter we improve these previous upper bounds, and generalize them to adelic fields satisfying the product formula, and to height functions associated to arbitrary nef toric metrized divisors.

Let $\Delta_0 \subset M_{\mathbb{R}}$ be the polytope defined by the toric Cartier divisor D_0. Following [19], we associate to $\overline{D}_0$ an adelic family of continuous concave functions $\vartheta_{0,v}\colon \Delta_0 \to \mathbb{R}$, $v \in \mathfrak{M}$, called the local roof functions of $\overline{D}_0$, see Proposition 2.3.28. For $i = 1, \dots, n$, write

$$f_i = \sum_{m\in M} \alpha_{i,m}\chi^m$$

with $\alpha_{i,m} \in \mathbb{K}$, and denote by Δ_i their corresponding Newton polytope. Let $N_{\mathbb{R}} = M_{\mathbb{R}}^{\vee} \simeq \mathbb{R}^n$ be the dual space and, for each place $v \in \mathfrak{M}$, consider the concave function $\psi_{i,v}\colon N_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\psi_{i,v}(u) = \begin{cases} -\log\Big(\displaystyle\sum_{m\in M} |\alpha_{i,m}|_v \,\mathrm{e}^{-\langle m,u\rangle}\Big) & \text{if } v \text{ is Archimedean,} \\ -\log\Big(\displaystyle\max_{m\in M} |\alpha_{i,m}|_v \,\mathrm{e}^{-\langle m,u\rangle}\Big) & \text{if } v \text{ is non-Archimedean.} \end{cases} \tag{2.1.2}$$

The Legendre-Fenchel dual

$$\vartheta_{i,v} = \psi_{i,v}^{\vee} = \inf_{u\in N_{\mathbb{R}}} \langle x,u\rangle - \psi_{i,v}(u)$$

is a continuous concave function on Δ_i. Furthermore, we denote by MI_M the mixed integral of a family of $n+1$ concave functions on convex bodies of $M_{\mathbb{R}}$ (Definition 2.3.30).

Then, the main result of this chapter (Theorem 2.4.5) gives the following upper bound

$$\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n)) \le \sum_{v\in\mathfrak{M}} n_v \,\mathrm{MI}(\vartheta_{0,v},\dots,\vartheta_{n,v}). \tag{2.1.3}$$

It's proof relies on the construction of nef toric metrized divisors $\overline{D}_i$ on a suitable toric variety, such that each f_i corresponds to a small section of $\overline{D}_i$. Indeed, they correspond to the concave functions in (2.1.2), see Proposition 2.4.2 and Lemma 2.4.4. Then, one proceeds by applying the constructions and results of [17, 19] and basic results from arithmetic intersection theory.

However, trying to keep a certain level of generality, we faced difficulties to define and study global heights of cycles over adelic fields. This lead us to a more detailed study of these notions. In particular, we detail a notion of adelic field extension that preserves the product formula (Definition 2.3.5), and a well-defined notion of global height for cycles with respect to metrized divisors that are generated by small sections (Proposition-Definition 2.3.22).

Using the basic properties of the mixed integral, we can bound the right-hand side of (2.1.3) in terms of mixed volumes. From this, we can derive the bound in Corollary 2.4.8:

$$\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n)) \le \mathrm{MV}_M(\Delta_1,\dots,\Delta_n)\Big(\sum_{v\in\mathfrak{M}} \max\vartheta_{0,v}\Big) + \sum_{i=1}^{n} \mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n)\ell(f_i), \quad (2.1.4)$$

where $\ell(f_i)$ denotes the (logarithmic) length of f_i, see Definition 2.4.6. This bound should be compared with the one given by the arithmetic Bézout theorem (Corollary 2.4.9), which follows as a direct consequence to these results. Inequality (2.1.4) gives a far more treatable bound than the one appearing in (2.1.3); however, there are cases in which the bounding of the mixed integrals by the length and mixed volume may proof inefficient, see Example 2.4.12.

The following illustrates a typical application of these results. It concerns two height functions applied to the same 0-cycle. Our upper bounds are close to optimal for both of them and, in particular, they reflect their very different behaviour on this family of Laurent polynomials. We refer to Example 2.4.11 for details.

Example 2.1.1. Take integers $d,\alpha\ge 1$ and consider the system of Laurent polynomials

$$f_1 = x_1-\alpha,\quad f_2 = x_2-\alpha x_1^d,\quad \dots,\quad f_n = x_n - \alpha x_{n-1}^d \quad \in \mathbb{Q}[x_1^{\pm1},\dots,x_n^{\pm1}].$$

The 0-cycle $Y := Z(f_1,\dots,f_n)$ of $\mathbb{G}^n_{\mathrm{m},\mathbb{Q}}$ is the single point $(\alpha,\alpha^{d+1},\dots,\alpha^{d^{n-1}+\dots+d+1})$ with multiplicity 1.

Let $\mathbb{P}^n_{\mathbb{Q}}$ be the n-dimensional projective space over $\mathbb{Q}$ and $\overline{E}^{\mathrm{can}}$ the divisor of the hyperplane at infinity, equipped with the canonical metric. Its associated height function is the Weil height. We consider two toric compactifications X_1 and X_2 of $\mathbb{G}^n_{\mathrm{m}}$. These are given by compactifying the torus via the equivariant embeddings $\iota_i\colon \mathbb{G}^n_{\mathrm{m}} \hookrightarrow \mathbb{P}^n_{\mathbb{Q}}$, $i=1,2$, respectively defined, for $p=(p_1,\dots,p_n)\in\mathbb{G}^n_{\mathrm{m}}(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^{\times})^n$, by

$$\iota_1(p) = (1:p_1:\dots:p_n) \quad\text{and}\quad \iota_2(p) = (1:p_1:p_2p_1^{-d}:\dots:p_np_{n-1}^{-d}).$$

Set $\overline{D}_i = \iota_i^*\,\overline{E}^{\mathrm{can}}$, which is a nef toric metrized divisor on X_i, $i=1,2$. By an explicit computation, we show that

$$\mathrm{h}_{\overline{D}_1}(Y) = \Big(\sum_{i=1}^{n} d^{i-1}\Big)\log(\alpha) \quad\text{and}\quad \mathrm{h}_{\overline{D}_2}(Y) = \log(\alpha).$$

On the other hand, the upper bounds given by (2.1.3) are

$$\mathrm{h}_{\overline{D}_1}(Y) \le \Big(\sum_{i=1}^{n} d^{i-1}\Big)\log(\alpha+1) \quad\text{and}\quad \mathrm{h}_{\overline{D}_2}(Y) \le n\log(\alpha+1).$$

As further application of (2.1.3), we give an upper bound for the size of the coefficients of the $\boldsymbol{u}$-resultant of the direct image under a monomial map of the solution set of a system of Laurent polynomial equations. The following version of this result is contained in the statement of Theorem 2.4.14.

For the simplicity of the exposition, set $\mathbb{K} = \mathbb{Q}$ and $M = \mathbb{Z}^n$. Let $r \geq 0$, $\boldsymbol{m}_0 = (m_{0,0}, \dots, m_{0,r}) \in (\mathbb{Z}^n)^{r+1}$ and $\boldsymbol{\alpha}_0 = (\alpha_{0,0}, \dots, \alpha_{0,r}) \in (\mathbb{Z} \setminus \{0\})^{r+1}$, and consider the map $\varphi_{\boldsymbol{m}_0,\boldsymbol{\alpha}_0} \colon \mathbb{G}_{\mathrm{m},\mathbb{Q}}^n \to \mathbb{P}_{\mathbb{Q}}^r$ defined by

$$\varphi_{\boldsymbol{m}_0,\boldsymbol{\alpha}_0}(p) = (\alpha_{0,0}\chi^{m_{0,0}}(p) : \cdots : \alpha_{0,r}\chi^{m_{0,r}}(p)). \tag{2.1.5}$$

For a 0-cycle W of $\mathbb{P}_{\mathbb{Q}}^r$, let $\boldsymbol{u} = (u_0, \dots, u_r)$ be a group of $r+1$ variables and denote by $\mathrm{Res}(W) \in \mathbb{Z}[u_0, \dots, u_r]$ its primitive $\boldsymbol{u}$-resultant, see Definition 2.4.13, which is well-defined up a sign.

Theorem 2.1.2. *Let $f_1, \dots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $\boldsymbol{m}_0 \in (\mathbb{Z}^n)^{r+1}$ and $\boldsymbol{\alpha}_0 \in (\mathbb{Z} \setminus \{0\})^{r+1}$ with $r \geq 0$. Set $\Delta_0 = \mathrm{conv}(m_{0,0}, \dots, m_{0,r}) \subset \mathbb{R}^n$ and let φ be the monomial map associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (2.1.5). For $i = 1, \dots, n$, let $\Delta_i \subset \mathbb{R}^n$ be the Newton polytope of f_i, and $\boldsymbol{\alpha}_i$ the vector of nonzero coefficients of f_i. Then*

$$\ell(\mathrm{Res}(\varphi_* Z(f_1, \dots, f_n))) \leq \sum_{i=0}^{n} \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)\, \ell(\boldsymbol{\alpha}_i),$$

where $\ell(\cdot)$ represents the logarithmic length.

2.2 The classical Bernštein-Kusnirenko theorem

In this section, we recall the proof of the Bernštein-Kušnirenko theorem using intersection theory on toric varieties, which is the model that we follow in our treatment of the arithmetic version of this result. Presenting this proof also allows us to introduce the basic definitions and results on the intersection of Cartier divisors with cycles, and on the algebraic geometry of toric varieties. For more details on these subjects, we refer to [33, 53] and to [34].

2.2.1 Intersection theory

Let K be an infinite field and X a variety over K of dimension n. For $0 \leq k \leq n$, the *group of k-cycles*, denoted by $Z_k(X)$, is the free abelian group on the k-dimensional irreducible subvarieties of X. Thus, a *k-cycle* is a finite formal sum

$$Y = \sum_V m_V V$$

where the V's are k-dimensional irreducible subvarieties of X and the m_V's are integers. The *support* of Y, denoted by $|Y|$, is the union of the subvarieties V such that $m_V \neq 0$.

The cycle Y is *effective* if $m_V \geq 0$ for every V. Given $Y, Y' \in Z_k(X)$, we write $Y' \leq Y$ whenever $Y - Y'$ is effective.

Let Z be a subscheme of X of pure dimension k. For an irreducible component V of Z, we denote by $\mathcal{O}_{V,Z}$ the local ring of Z along V, and by $l_{\mathcal{O}_{V,Z}}(\mathcal{O}_{V,Z})$ its length as an $\mathcal{O}_{V,Z}$-module. The k-cycle associated to Z is then defined as

$$[Z] = \sum l_{\mathcal{O}_{V,Z}}(\mathcal{O}_{V,Z})\, V,$$

the sum being over the irreducible components of Z.

Let V be an irreducible subvariety of X of codimension one and f a regular function on an open subset U of X such that $U \cap V \neq \emptyset$. The *order of vanishing* of f along V is defined as

$$\mathrm{ord}_V(f) = l_{\mathcal{O}_{V,X}(U)}(\mathcal{O}_{V,X}(U)/(f)).$$

For a Cartier divisor D on X, the *order of vanishing* of D along V is defined as

$$\mathrm{ord}_V(D) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h)$$

with $g, h \in \mathcal{O}_{V,X}(U)$ such that g/h is a local equation of D on an open subset U of X with $U \cap V \neq \emptyset$. This definition does not depend on the choice of U, g and h. Moreover, $\mathrm{ord}_V(D) = 0$ for all but a finite number of V's. The Weil divisor associated to D is then defined as

$$D \cdot X = \sum_V \mathrm{ord}_V(D)\, V, \tag{2.2.1}$$

the sum being over all irreducible subvarieties of X of codimension one. The *support* of D, denoted by $|D|$, is the support of $D \cdot X$.

Now let W be an irreducible subvariety of X of dimension k. If $W \not\subset |D|$, then D restricts to a Cartier divisor on W. In this case, we define $D \cdot W$ as the Weil divisor of W obtained by restricting (2.2.1) to W. This gives a $(k-1)$-cycle of X. If $W \subset |D|$, then we set $D \cdot W = 0$, the zero element of $Z_{k-1}(X)$. We extend by linearity this intersection product to a morphism

$$Z_k(X) \longrightarrow Z_{k-1}(X), \quad Y \longmapsto D \cdot Y,$$

with the convention that $Z_{-1}(X) = 0$, the zero group.

For $0 \leq r \leq n$ and Cartier divisors D_i on X, $i = 1, \dots, r$, we define inductively the intersection product $\prod_{i=1}^r D_i \in Z_{n-r}(X)$ by

$$\prod_{i=1}^t D_i = \begin{cases} X & \text{if } t = 0, \\ D_1 \cdot \prod_{i=2}^t D_i & \text{if } 1 \leq t \leq r. \end{cases}$$

Definition 2.2.1. Let Y be a k-cycle of X and $D_1, \dots, D_r$ Cartier divisors on X, with $r \leq k$. We say that $D_1, \dots, D_r$ *intersect* Y *properly* if, for every subset $I \subset \{1, \dots, r\}$,

$$\dim\Big(|Y| \cap \bigcap_{i \in I} |D_i|\Big) = k - \#I.$$

If $D_1, \dots, D_r$ intersect X properly, then the cycle $\prod_{i=1}^r D_i$ does not depend on the order of the D_i's. We refer to [33, Corollary 2.4.2] for a proof of this statement in the case of pseudo-divisors, which is a generalization of Cartier divisors. This conclusion does not necessarily hold if these divisors do not intersect properly.

Example 2.2.2. Let $X = \mathbb{A}_K^2$ and consider the principal Cartier divisors $D_1 = \mathrm{div}(x_1x_2)$ and $D_2 = \mathrm{div}(x_1)$ given by taking all local equations equal to x_1x_2 and x_1, respectively. Then

$$D_1 \cdot D_2 = 0 \quad \text{and} \quad D_2 \cdot D_1 = (0,0).$$

Proposition 2.2.3. *Let X be an equidimensional Cohen-Macaulay variety over K of dimension n, and $D_1, \dots, D_n$ Cartier divisors on X. Let s_i be a global section of $\mathcal{O}(D_i)$, $i = 1, \dots, n$, and write*

$$\prod_{i=1}^n \mathrm{div}(s_i) = \sum_p m_p\, p \in Z_0(X), \tag{2.2.2}$$

where the sum is over the closed points p of X and $m_p \in \mathbb{Z}$. This 0-cycle is effective and, for each isolated closed point p of the intersection $\bigcap_{i=1}^n |\mathrm{div}(s_i)|$,

$$m_p = \dim_K(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)),$$

where U is a trivializing neighborhood of p, and f_i is a defining function for s_i on U, $i = 1, \dots, n$.

Proof. The fact that the cycle in (2.2.2) is effective follows from the hypothesis that the s_i's are global sections.

For the second statement, by possibly replacing U with a smaller open neighborhood of p, we can assume that $\mathrm{div}(s_1), \dots, \mathrm{div}(s_n)$ intersect X properly on U. So, by Definition 2.2.1, this intersection on U is of dimension 0. By [33, Proposition 7.1 and Example 7.1.10],

$$m_p = l_{\mathcal{O}_{p,X}(U)}(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)).$$

By [33, Lemma A.1.3 and Example A.1.1], we have the equality

$$l_{\mathcal{O}_{p,X}(U)}(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)) = \dim_K(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)),$$

completing the proof. $\square$

For the rest of this section, we assume that the variety X is projective. With this hypothesis, Chow's moving lemma allows to construct, given a cycle and a family of Cartier divisors, another family of linearly equivalent Cartier divisors intersecting the given cycle properly, in the sense of Definition 2.2.1.

Definition 2.2.4. Let Y be a k-cycle of X and $D_1, \ldots, D_k$ Cartier divisors on X. The *degree* of Y with respect to $D_1, \ldots, D_k$, denoted by $\deg_{D_1,\ldots,D_k}(Y)$, is inductively defined by the rules:

1. if $k = 0$, write $Y = \sum_p m_p\, p$, and set $\deg(Y) = \sum_p m_p\, [\mathrm{K}(p) : K]$;
2. if $k \geq 1$, choose a rational section s_k of $\mathcal{O}(D_k)$ such that $\mathrm{div}(s_k)$ intersects Y properly, and set $\deg_{D_1,\ldots,D_k}(Y) = \deg_{D_1,\ldots,D_{k-1}}(\mathrm{div}(s_k) \cdot Y)$.

The degree of a cycle with respect to a family of Cartier divisors does not depend on the choice of the rational section s_k in 2, see for instance [33, § 2.5] or [53, § 1.1.C].

A Cartier divisor D on X is *nef* if $\deg_D(C) \geq 0$ for every irreducible curve C of X. By Kleiman's theorem [53, §1.4.B], for a family of nef Cartier divisors $D_1, \ldots, D_k$ on X and an effective k-cycle Y of X,

$$\deg_{D_1,\ldots,D_k}(Y) \geq 0. \tag{2.2.3}$$

Proposition 2.2.5. *Let Y be an effective k-cycle of X and $D_1, \ldots, D_k$ nef Cartier divisors on X. Let s_k be a global section of $\mathcal{O}(D_k)$. Then*

$$0 \leq \deg_{D_1,\ldots,D_{k-1}}(\mathrm{div}(s_k) \cdot Y) \leq \deg_{D_1,\ldots,D_k}(Y).$$

Proof. Since Y is effective and s_k is a global section, $\mathrm{div}(s_k) \cdot Y$ is also effective. Since $D_1, \ldots, D_{k-1}$ are nef, by (2.2.3) we have that $\deg_{D_1,\ldots,D_{k-1}}(\mathrm{div}(s_k) \cdot Y) \geq 0$, proving the lower bound.

For the upper bound, we reduce without loss of generality to the case when $Y = V$ is an irreducible subvariety of dimension k. If $V \subset |\,\mathrm{div}(s_k)|$, then $\mathrm{div}(s_k)\cdot Y = 0 \in Z_{k-1}(X)$. Hence $\deg(\mathrm{div}(s_k) \cdot Y) = 0$ and the bound follows from the nefness of the D_i's. Otherwise, from the definition of the degree,

$$\deg_{D_1,\ldots,D_{k-1}}(\mathrm{div}(s_k) \cdot V) = \deg_{D_1,\ldots,D_k}(V),$$

which completes the proof. $\square$

Corollary 2.2.6. *Let $D_1, \ldots, D_n$ be nef Cartier divisors on X and, for $i = 1, \ldots, n$, let s_i be a global section of $\mathcal{O}(D_i)$. Then*

$$0 \leq \deg\left(\prod_{i=1}^{n} \mathrm{div}(s_i)\right) \leq \deg_{D_1,\ldots,D_n}(X).$$

2.2.2 Toric varieties

Let $M \simeq \mathbb{Z}^n$ be a lattice, and set

$$K[M] \simeq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \text{and} \quad \mathbb{T} = \operatorname{Spec}(K[M]) \simeq \mathbb{G}_{\mathrm{m},K}^n \tag{2.2.4}$$

for its group K-algebra and algebraic torus over K, respectively. The elements of M correspond to the characters of $\mathbb{T}$ and, given $m \in M$, we denote by $\chi^m \in \operatorname{Hom}(\mathbb{T}, \mathbb{G}_{\mathrm{m},K})$ the corresponding character. Set also $M_{\mathbb{R}} = M \otimes \mathbb{R}$

Let $N = M^\vee \simeq \mathbb{Z}^n$ be the dual lattice and set $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Given a complete fan Σ in $N_{\mathbb{R}}$, we denote by X_Σ the associated toric variety with torus $\mathbb{T}$. It is a proper normal variety over K containing $\mathbb{T}$ as a dense open subset and such that the action of $\mathbb{T}$ on itself extends to X_Σ. When the fan Σ is *regular*, in the sense that it is induced by a piecewise linear concave function on $N_{\mathbb{R}}$, the toric variety X_Σ is projective.

Let $X = X_\Sigma$ be a toric variety, and D be a toric Cartier divisor on X, that is a $\mathbb{T}$-invariant Cartier divisor. We denote by Ψ_D its associated virtual support function on Σ. This is a piecewise linear function $\Psi_D \colon N_{\mathbb{R}} \to \mathbb{R}$ satisfying that, for each cone $\sigma \in \Sigma$, there exists $m \in M$ such that, for all $u \in \sigma$,

$$\Psi_D(u) = \langle m, u \rangle.$$

The condition that Ψ_D is concave is both equivalent to the conditions that D is nef and that the line bundle $\mathcal{O}(D)$ is globally generated. This line bundle $\mathcal{O}(D)$ is a subsheaf of the sheaf of rational functions of X. For each $m \in M$, the character χ^m is a rational function of X, and so it induces a rational section of $\mathcal{O}(D)$ that is regular and nowhere vanishing on $\mathbb{T}$. The rational section corresponding to the point $m = 0$ is called the *distinguished rational section* of $\mathcal{O}(D)$ and denoted by s_D.

The toric Cartier divisor D also determines the lattice polytope of $M_{\mathbb{R}}$ given by

$$\Delta_D = \{x \in M_{\mathbb{R}} \mid \ \langle x, u \rangle \geq \Psi_D(u) \text{ for every } u \in N_{\mathbb{R}}\}.$$

A rational section corresponding to a point $m \in M$ is global if and only if $m \in \Delta_D$. The global sections corresponding to the lattice points of Δ_D form a K-basis for the space of global sections of $\mathcal{O}(D)$. Identifying each character χ^m with the corresponding rational section ς_m of $\mathcal{O}(D)$, we have the decomposition

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{m \in \Delta_D \cap M} K \cdot \varsigma_m. \tag{2.2.5}$$

Now let $\Delta_1, \dots, \Delta_r$ be lattice polytopes in $M_{\mathbb{R}}$. For each Δ_i, we consider its *support function*, which is the piecewise linear concave function with lattice slopes $\Psi_{\Delta_i} \colon N_{\mathbb{R}} \to \mathbb{R}$ given by

$$\Psi_{\Delta_i}(u) = \min_{x \in \Delta_i} \langle x, u \rangle. \tag{2.2.6}$$

Let Σ be a regular complete fan in $N_{\mathbb{R}}$ compatible with the collection $\Delta_1, \dots, \Delta_r$, in the sense that the Ψ_{Δ_i}'s are virtual support functions on Σ. Such a fan can be constructed by taking any regular complete fan in $N_{\mathbb{R}}$ refining the complex of cones that are normal to the faces of Δ_i, for all i. Let X be the toric variety corresponding to this fan and D_i the toric Cartier divisor on X corresponding to these virtual support functions. By construction, Ψ_{Δ_i} is concave. Hence D_i is nef and $\mathcal{O}(D_i)$ is globally generated, and its associated polytope coincides with Δ_i.

Definition 2.2.7. The *mixed volume* of $\Delta_1, \dots, \Delta_n$ is defined as the alternating sum

$$\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \le i_1 < \dots < i_j \le n} \mathrm{vol}_M(\Delta_{i_1} + \dots + \Delta_{i_j}),$$

where vol_M be the Haar measure on $M_{\mathbb{R}}$ such that M has covolume 1, and take $r = n$.

A fundamental result in toric geometry states that the degree of a toric variety with respect to a family of nef toric Cartier divisors is given by the mixed volume of its polytopes [34, § 5.4]. In our present setting, this amounts to the formula

$$\deg_{D_1, \dots, D_n}(X) = \mathrm{MV}_M(\Delta_1, \dots, \Delta_n). \tag{2.2.7}$$

2.2.3 The Bernštein-Kušnirenko theorem

We first associate a 0-cycle of the torus to a family of Laurent polynomials on M.

Definition 2.2.8. Let $f_1, \dots, f_n \in K[M]$, and denote by $V(f_1, \dots, f_n)_0$ the set of isolated closed points in the variety defined by this family of Laurent polynomials. For each $p \in V(f_1, \dots, f_n)_0$, let $\mathfrak{m}_p$ be the maximal ideal of $K[M]$ corresponding to p and set

$$\mu_p = \dim_K(K[M]_{\mathfrak{m}_p}/(f_1, \dots, f_n)).$$

The *0-cycle associated to* $f_1, \dots, f_n$ is defined as

$$Z(f_1, \dots, f_n) = \sum_{p \in V(f_1, \dots, f_n)_0} \mu_p \, p \in Z_0(\mathbb{T}).$$

Let $f = \sum_{m \in M} \alpha_m \chi^m \in K[M]$ be a Laurent polynomial. Its *support* is defined as the finite subset of M of the exponents of its nonzero terms, that is $\mathrm{supp}(f) = \{m \mid \alpha_m \neq 0\}$. The *Newton polytope* of f is the lattice polytope in $M_{\mathbb{R}}$ given by the convex hull of its support, that is $\mathcal{N}(f) = \mathrm{conv}(\mathrm{supp}(f))$.

The following proposition gives us the relation between the 0-cycle in Definition 2.2.8 and the one arising from intersection theory.

Proposition 2.2.9. *Let $f_1, \dots, f_n \in K[M]$. Let Σ be a regular complete fan in $N_{\mathbb{R}}$ compatible with the Newton polytopes of the f_i's. For $i = 1, \dots, n$, let D_i be the Cartier divisor on X_Σ associated to $\mathcal{N}(f_i)$, and s_i the global section of $\mathcal{O}(D_i)$ corresponding to f_i as in* (2.2.5). *Write*

$$\prod_{i=1}^{n} \operatorname{div}(s_i) = \sum_p \nu_p \, p,$$

where the sum is over all closed points p of X_Σ and $\nu_p \in \mathbb{Z}$. Then

1. *for every $p \in V(f_1, \dots, f_n)_0$, we have $\nu_p = \dim_K(K[M]_{\mathfrak{m}_p}/(f_1, \dots, f_n))$;*

2. *the inequality $Z(f_1, \dots, f_n) \leq \prod_{i=1}^{n} \operatorname{div}(s_i)$ holds.*

Proof. We have that $\bigcap_{i=1}^{n} |\operatorname{div}(s_i)| = V(f_1, \dots, f_n)$. Since $\mathbb{T}$ is Cohen-Macaulay, Proposition 2.2.3 gives the first statement. Since the sections s_i are global, the 0-cycle $\prod_{i=1}^{n} \operatorname{div}(s_i)$ is effective. Hence, the second statement follows directly from the first one. □

We conclude this section by proving the version of the Bernštein-Kušnirenko theorem as presented in (2.1.1).

Theorem 2.2.10. *Let $f_1, \dots, f_n \in K[M]$ be a family of Laurent polynomials, and let Δ_i denote the newton polytope of f_i, for every $i = 1, \dots, n$. Then*

$$\deg(Z(f_1, \dots, f_n)) \leq \mathrm{MV}_M(\Delta_1, \dots, \Delta_n).$$

Proof. This follows from Proposition 2.2.9(2), Corollary 2.2.6 and the formula (2.2.7). □

Remark. *It should be noted that, for a fixed family of convex polytopes $\Delta_1, \dots, \Delta_n \subset M_{\mathbb{R}}$ with integer vertices, and for generic Laurent polynomials supported on these Δ_i's, Bernštein-Kušnirenko's theorem gives in fact an equality.*

2.3 Arithmetic of toric varieties

In this section we consider adelic fields following [19], and give a detailed construction of adelic field extension that preserves the product formula. In this sense it is an extension of the one in *loc. cit.*, which was only meant to preserve the product formula when dealing with extensions of number fields and function fields of curves. We then introduce a notion of global height for cycles of a variety over a such field, giving an explicit description of this construction in the 0-dimensional case. Finally, we recall the necessary background on the arithmetic geometry of toric varieties. We refer to [17, 19] for more details.

2.3.1 Adelic fields and finite extensions

We first introduce the notion of arithmetic field on which we give our results.

Definition 2.3.1. Let $\mathbb{K}$ be an infinite field and $\mathfrak{M}$ a set of places. Each place $v \in \mathfrak{M}$ is a pair consisting of an absolute value $|\cdot|_v$ and a positive real weight n_v. We say that $(\mathbb{K}, \mathfrak{M})$ is an *adelic field* if

1. for each $v \in \mathfrak{M}$, the absolute value $|\cdot|_v$ is either Archimedean or associated to a nontrivial discrete valuation;
2. for each $\alpha \in \mathbb{K}^\times$, we have that $|\alpha|_v = 1$ for all but a finite number of $v \in \mathfrak{M}$.

Moreover, we say that an adelic field $(\mathbb{K}, \mathfrak{M})$ satisfies the *product formula* if

$$\prod_{v \in \mathfrak{M}} |\alpha|_v^{n_v} = 1,$$

for every $\alpha \in \mathbb{K}^\times$.

Example 2.3.2. Let $\mathfrak{M}_\mathbb{Q}$ be the set of places of $\mathbb{Q}$ consisting of the Archimedean and p-adic absolute values of $\mathbb{Q}$, normalized in the standard way, and with all the weights equal to 1. The adelic field $(\mathbb{Q}, \mathfrak{M}_\mathbb{Q})$ satisfies the product formula.

Example 2.3.3. Let $\mathrm{K}(C)$ denote the function field of a regular projective curve C over a field κ. To each closed point $v \in C$ we associate the absolute value and weight given, for a non-zero element $f \in \mathrm{K}(C)$, by

$$|f|_v = c_\kappa^{-\operatorname{ord}_v(f)} \quad \text{and} \quad n_v = [\mathrm{K}(v) : \kappa], \tag{2.3.1}$$

where $\operatorname{ord}_v(f)$ denotes the order of vanishing of f at v and

$$c_\kappa = \begin{cases} \mathrm{e} & \text{if } \#\kappa = \infty, \\ \#\kappa & \text{if } \#\kappa < \infty. \end{cases} \tag{2.3.2}$$

The set of places $\mathfrak{M}_{\mathrm{K}(C)}$ is indexed by the closed points of C, and consists of these absolute values and weights. The pair $(\mathrm{K}(C), \mathfrak{M}_{\mathrm{K}(C)})$ is an adelic field which satisfies the product formula.

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field. For each place $v \in \mathfrak{M}$, we denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to the absolute value $|\cdot|_v$. By a theorem of Ostrowski, if v is Archimedean, then $\mathbb{K}_v$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ [21, Chapter 3, Theorem 1.1]. In particular, an adelic field has only a finite number of Archimedean places.

Lemma 2.3.4. *Let $\mathbb{F}$ be a finite extension of $\mathbb{K}$ and $v \in \mathfrak{M}$. Then*

$$\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v \simeq \bigoplus_w E_w, \tag{2.3.3}$$

where the sum is over the absolute values $|\cdot|_w$ on $\mathbb{F}$ whose restriction to $\mathbb{K}_v$ coincides with $|\cdot|_v$, and where the E_w's are local Artinian $\mathbb{K}_v$-algebras with maximal ideal $\mathfrak{p}_w$. For each w, we have $E_w/\mathfrak{p}_w \simeq \mathbb{F}_w$.

Proof. Since $\mathbb{K} \hookrightarrow \mathbb{F}$ is a finite extension, the tensor product $\mathbb{F} \otimes \mathbb{K}_v$ is an Artinian $\mathbb{K}_v$-algebra. By the structure theorem for Artinian algebras,

$$\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v \simeq \bigoplus_{i \in I} E_i,$$

where I is a finite set and the E_i's are local Artinian $\mathbb{K}_v$-algebras. Let $\mathfrak{p}_i$ be the maximal ideal of E_i, for each i. These are the only prime ideals of $\mathbb{F} \otimes \mathbb{K}_v$, and so $\mathrm{rad}(\mathbb{F} \otimes \mathbb{K}_v) = \bigcap_{i \in I} \mathfrak{p}_i$.

Each w in the decomposition (2.3.3) corresponds to an absolute value $|\cdot|_w$ on $\mathbb{F}$ extending $|\cdot|_v$, and there is a natural inclusion $\mathbb{F} \hookrightarrow \mathbb{F}_w$. The diagonal morphism $\mathbb{F} \to \bigoplus_w \mathbb{F}_w$ extends to a map of $\mathbb{K}_v$-vector spaces

$$\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v \longrightarrow \bigoplus_w \mathbb{F}_w.$$

By [13, Chapitre VI, §8.2 Proposition 11(b)], this morphism is surjective and its kernel is the radical ideal of $\mathbb{F} \otimes \mathbb{K}_v$. Therefore

$$\bigoplus_{i \in I} E_i/\mathfrak{p}_i = \Big(\bigoplus_{i \in I} E_i \Big) \Big/ \mathrm{rad}(\mathbb{F} \otimes \mathbb{K}_v) \simeq \bigoplus_w \mathbb{F}_w. \tag{2.3.4}$$

The summands in both extremes of (2.3.4) are fields over $\mathbb{K}_v$, and so Artinian local $\mathbb{K}_v$-algebras. By the uniqueness of the decomposition in the structure theorem for Artinian algebras, there is a bijection between the elements in I and the w's, identifying each $i \in I$ with the unique w such that $E_i/\mathfrak{p}_i \simeq \mathbb{F}_w$. □

The following definition for adelic field extension is equivalent to the one proposed by Gubler for M-fields, see [39, Remark 2.5].

Definition 2.3.5. Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field and $\mathbb{F}$ a finite extension of $\mathbb{K}$. For every place $v \in \mathfrak{M}$, we denote by $\mathfrak{N}_v$ the set of absolute values $|\cdot|_w$ on $\mathbb{F}$ that extend $|\cdot|_v$ with weight given by

$$n_w = \frac{\dim_{\mathbb{K}_v}(E_w)}{[\mathbb{F} : \mathbb{K}]} n_v,$$

where the E_w's are the Artinian $\mathbb{K}_v$-algebras in the decomposition of $\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v$ from Lemma 2.3.4. Set $\mathfrak{N} = \bigsqcup_{v \in \mathfrak{M}} \mathfrak{N}_v$. The pair $(\mathbb{F}, \mathfrak{N})$ is an adelic field. The adelic fields of this form are called *adelic field extensions* of $(\mathbb{K}, \mathfrak{M})$.

Remark. *With notation as in Lemma 2.3.4,*

$$\dim_{\mathbb{K}_v}(E_w) = l_{E_w}(E_w)[\mathbb{F}_w : \mathbb{K}_v],$$

where $l_{E_w}(E_w)$ is the length of E_w as a module over itself. This follows from [33, Lemma A.1.3] applied to the morphism $\mathbb{K}_v \to E_w$. Hence, the weights in Definition 2.3.5 can be alternatively written as

$$n_w = l_{E_w}(E_w)\frac{[\mathbb{F}_w : \mathbb{K}_v]}{[\mathbb{F} : \mathbb{K}]} n_v.$$

Proposition 2.3.6. *Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field and $(\mathbb{F}, \mathfrak{N})$ an adelic field extension of $(\mathbb{K}, \mathfrak{M})$. Then*

1. *the equality $\sum_{w\in\mathfrak{N}_v} n_w = n_v$ holds for every place $v \in \mathfrak{M}$;*
2. *if $(\mathbb{K}, \mathfrak{M})$ satisfies the product formula, then $(\mathbb{F}, \mathfrak{N})$ also does.*

Proof. From the definition of adelic field extension and Lemma 2.3.4,

$$\sum_{w\in\mathfrak{N}_v} n_w = \sum_{w\in\mathfrak{N}_v} \frac{\dim_{\mathbb{K}_v}(E_w)}{[\mathbb{F} : \mathbb{K}]} n_v = \frac{\dim_{\mathbb{K}_v}(\mathbb{F}\otimes\mathbb{K}_v)}{[\mathbb{F} : \mathbb{K}]} n_v = n_v,$$

which proves statement (1). To prove the second statement, let $\alpha \in \mathbb{F}^\times$ and consider the multiplication map $\eta_\alpha \colon \mathbb{F} \to \mathbb{F}$ given by $\eta_\alpha(x) = \alpha x$. The norm $N_{\mathbb{F}/\mathbb{K}}(\alpha) \in \mathbb{K}^\times$ is defined as the determinant of this $\mathbb{K}$-linear map. Moreover, η_α extends to the $\mathbb{K}_v$-linear map

$$\eta_\alpha \otimes 1_{\mathbb{K}_v} \colon \mathbb{F}\otimes\mathbb{K}_v \longrightarrow \mathbb{F}\otimes\mathbb{K}_v,$$

which has the same determinant. Using the decomposition in (2.3.3), write $\alpha\otimes 1_{\mathbb{K}_v} = (\alpha_w)_w$ with $\alpha_w \in E_w$. Hence $\eta_\alpha \otimes 1_{\mathbb{K}_v} = \bigoplus_w \eta_{\alpha_w}$ and

$$N_{\mathbb{F}/\mathbb{K}}(\alpha) = \det(\eta_\alpha \otimes 1_{\mathbb{K}_v}) = \prod_{w\in\mathfrak{N}_v} N_{E_w/\mathbb{K}_v}(\alpha_w).$$

By [14, Chapitre III, §9.2, Proposition 1], $N_{E_w/\mathbb{K}_v}(\alpha_w) = N_{\mathbb{F}_w/\mathbb{K}_v}(\alpha_w)^{l_{E_w}(E_w)}$. Moreover, by [50, VI Proposition 5.6],

$$N_{\mathbb{F}_w/\mathbb{K}_v}(\alpha_w) = \prod_\sigma \sigma(\alpha_w)^{[\mathbb{F}_w:\mathbb{K}_v]_i},$$

where the product is over the different embeddings σ of $\mathbb{F}_w$ in an algebraic closure of $\mathbb{K}_v$, and $[\mathbb{F}_w : \mathbb{K}_v]_i$ denotes the inseparability degree of the extension $\mathbb{K}_v \hookrightarrow \mathbb{F}_w$. Furthermore, the number of such embeddings is equal to the separability degree $[\mathbb{F}_w : \mathbb{K}_v]_s$. For every embedding σ, we have $|\sigma(\alpha_w)|_v = |\alpha|_w$ because the base field $\mathbb{K}_v$ is complete. Since $[\mathbb{F}_w : \mathbb{K}_v]_i[\mathbb{F}_w : \mathbb{K}_v]_s = [\mathbb{F}_w : \mathbb{K}_v]$, we get

$$|N_{\mathbb{F}/\mathbb{K}}(\alpha)|_v^{n_v} = \prod_{w\in\mathfrak{N}_v} |\sigma(\alpha_w)|_v^{l_{E_w}(E_w)[\mathbb{F}_w:\mathbb{K}_v]n_v} = \prod_{w\in\mathfrak{N}_v} |\alpha|_w^{[\mathbb{F}:\mathbb{K}]n_w}.$$

Since $N_{\mathbb{F}/\mathbb{K}}(\alpha) \in \mathbb{K}^\times$, if $(\mathbb{K}, \mathfrak{M})$ satisfies the product formula, then

$$\prod_{w\in\mathfrak{N}} |\alpha|_w^{n_w} = \left(\prod_{v\in\mathfrak{M}} |N_{\mathbb{F}/\mathbb{K}}(\alpha)|_v^{n_v}\right)^{\frac{1}{[\mathbb{F}:\mathbb{K}]}} = 1,$$

concluding the proof. □

Example 2.3.7. Let $\mathbb{F}$ be a number field. This is a separable extension of $\mathbb{Q}$. By [13, Chapitre VI, §8.5, Corollaire 3], we have that $\mathbb{F}\otimes\mathbb{Q}_v \simeq \bigoplus_{w\in\mathfrak{N}_v}\mathbb{F}_w$ for all $v\in\mathfrak{M}_{\mathbb{Q}}$. Therefore, the weight associated to each place $w\in\mathfrak{N}_v$ is

$$n_w = \frac{[\mathbb{F}_w : \mathbb{Q}_v]}{[\mathbb{F}:\mathbb{Q}]}.$$

Example 2.3.8. Let $(\mathrm{K}(C), \mathfrak{M}_{\mathrm{K}(C)})$ be the function field of a regular projective curve C over a field κ with the structure of adelic field as in Example 2.3.3. The places of $\mathrm{K}(C)$ correspond to the closed points of C with absolute values and weights given by (2.3.1). Let $\mathbb{F}$ be a finite extension of $\mathrm{K}(C)$ and $\mathfrak{N}$ the set of places of $\mathbb{F}$ as in Definition 2.3.5. There is a regular projective curve B over κ and a finite map $\pi\colon B\to C$ such that the extension $\mathrm{K}(C)\hookrightarrow\mathbb{F}$ identifies with the morphism $\pi^*\colon \mathrm{K}(C)\hookrightarrow\mathrm{K}(B)$. For each place $v\in\mathfrak{M}_{\mathrm{K}(C)}$, the absolute values of $\mathbb{F}$ that extend $|\cdot|_v$ are in bijection with the fiber $\pi^{-1}(v)$.

For a closed point $v\in C$, the integral closure in $\mathrm{K}(B)$ of $\mathcal{O}_{v,C}$ coincides with $\mathcal{O}_{\pi^{-1}(v),B}$, the local ring of B along the fibre $\pi^{-1}(v)$. The ring $\mathcal{O}_{\pi^{-1}(v),B}$ is of finite type over $\mathcal{O}_{v,C}$. With notation as in Lemma 2.3.4, by [13, Chapter VI, §8.5, Corollaire 3], we have $E_w\simeq\mathbb{F}_w$ for all $w\in\mathfrak{N}_v$. Hence, the weight of w is given by

$$n_w = \frac{[\mathbb{F}_w : \mathrm{K}(C)_v]}{[\mathbb{F}:\mathrm{K}(C)]}[\mathrm{K}(v):\kappa].$$

Let $e(w/v)$ denote the ramification index of w over v. By [13, Chapter VI, §8.5, Corollaire 2], we have that $[\mathbb{F}_w : \mathrm{K}(C)_v] = e(w/v)\,[\mathrm{K}(w):\mathrm{K}(v)]$. Therefore, for each place $w\in\mathfrak{N}_v$, the weight of w can also be expressed as

$$n_w = \frac{e(w/v)\,[\mathrm{K}(w):\kappa]}{[\mathbb{F}:\mathrm{K}(C)]}.$$

Following [19], a *global field* is a finite extension of the field of rational numbers or of the function field of a regular projective curve, with the structure of adelic field described in Examples 2.3.7 and 2.3.8. The discussions in these examples shows that this structure of adelic field extension coincides with the one given by Definition 2.3.5. In the case of function fields, it should be noted that the adelic structure depends on the extension.

Function fields of varieties of higher dimension provide examples of adelic fields satisfying the product formula, that are not global fields.

Example 2.3.9. Let $\mathrm{K}(S)$ be the function field of an irreducible normal variety S over a field κ of dimension $s \geq 1$, and $E_1, \dots, E_{s-1}$ nef Cartier divisors on S. Set $S^{(1)}$ for the set of irreducible hypersurfaces of S. For each $V \in S^{(1)}$, the local ring $\mathcal{O}_{V,S}$ is a discrete valuation ring. We associate to V the absolute value and weight given, for $f \in \mathrm{K}(S)$, by

$$|f|_V = c_\kappa^{-\operatorname{ord}_V(f)} n_v = \deg_{E_1,\dots,E_{s-1}}(V),$$

with c_κ as in (2.3.2). The set of places $\mathfrak{M}_{\mathrm{K}(S)}$ is indexed by $S^{(1)}$, and consists of these absolute values and weights. For $f \in \mathrm{K}(S)^\times$,

$$\sum_{V \in S^{(1)}} n_V \log |f|_v = \log(c_k) \sum_{V \in S^{(1)}} \deg_{E_1,\dots,E_{s-1}}(V) \operatorname{ord}_V(f) = \deg_{E_1,\dots,E_{s-1}}(\operatorname{div}(f)) = 0,$$

because the Cartier divisor $\operatorname{div}(f)$ is principal. Hence $(\mathrm{K}(S), \mathfrak{M}_{\mathrm{K}(S)})$ satisfies the product formula.

2.3.2 Height of cycles

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula, and X a normal projective variety over $\mathbb{K}$. For each place $v \in \mathfrak{M}$, we denote by X_v^{an} the v-adic analytification of X. In the Archimedean case, if $\mathbb{K}_v \simeq \mathbb{C}$, then X_v^{an} is an analytic space over $\mathbb{C}$ whereas, if $\mathbb{K}_v \simeq \mathbb{R}$, then X_v^{an} is an analytic space over $\mathbb{R}$, that is, an analytic space over $\mathbb{C}$ together with an antilinear involution, as explained in [19, Remark 1.1.5]. In the non-Archimedean case, X_v^{an} is a Berkovich space over $\mathbb{K}_v$ as in [19, § 1.2].

Fix $v \in \mathfrak{M}$ and set

$$X_v = X \times \operatorname{Spec}(\mathbb{K}_v).$$

Given a 0-cycle Y of X_v, a usual construction in Arakelov geometry associates a signed measure on X_v^{an}, denoted by δ_Y, that is supported on $|Y|^{\mathrm{an}}$ and has total mass equal to $\deg(Y)$, see for instance [19, Definition 1.3.15] for the non-Archimedean case. In what follows, we explicit this construction.

Let q be a closed point of X_v. The function field $\mathrm{K}(q)$ is a finite extension of $\mathbb{K}_v$ and $\deg(q) = [\mathrm{K}(q) : \mathbb{K}_v]$. If v is Archimedean, then $\deg(q)$ is either equal to 1 or 2. In the first case, the analytification of q is a point of X_v^{an} whereas, in the second case, it is a pair of conjugate points. If v is non-Archimedean, choose an affine open neighborhood $U = \operatorname{Spec}(A)$ of q and $A \to \mathbb{K}_v$ the corresponding morphism of $\mathbb{K}_v$-algebras. The analytification of q is the point $q^{\mathrm{an}} \in U^{\mathrm{an}} \subset X_v^{\mathrm{an}}$ corresponding to the multiplicative seminorm given by the composition

$$A \longrightarrow \mathrm{K}(q) \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0},$$

where $|\cdot|$ is the unique extension to $\mathrm{K}(q)$ of the absolute value $|\cdot|_v$.

Since the measure δ_q is supported on the point q^{an} and has total mass $\deg(q)$, it follows that

$$\delta_q = [\mathrm{K}(q) : \mathbb{K}_v]\, \delta_{q^{\mathrm{an}}}, \tag{2.3.5}$$

where $\delta_{q^{\mathrm{an}}}$ denotes the Dirac delta measure on q^{an}. For an arbitrary 0-cycle Y of X_v, the signed measure δ_Y is obtained from (2.3.5) by linearity. It is discrete signed measure of total mass equal to $\deg(Y)$.

Let D be a Cartier divisor on X. A *metric* on the analytic line bundle $\mathcal{O}(D)_v^{\mathrm{an}}$ is an assignment that, to each open subset $U \subset X_v^{\mathrm{an}}$ and local section s on U, associates a continuous function

$$\|s(\cdot)\|_v \colon U \longrightarrow \mathbb{R}_{\geq 0}$$

that is compatible with restrictions to open subsets, vanishes only when the local section does, and respects multiplication of local sections by analytic functions, see [19, Definitions 1.1.1 and 1.3.1]. This notion allows to define local heights of 0-cycles.

Definition 2.3.10. Let D be a Cartier divisor on X, and $\|\cdot\|_v$ a metric on $\mathcal{O}(D)_v^{\mathrm{an}}$. For a 0-cycle Y of X_v and a rational section s of $\mathcal{O}(D)$ that is regular and non-vanishing on the support of Y, the *local height* of Y with respect to the pair $(\|\cdot\|_v, s)$ is defined as

$$\mathrm{h}_{\|\cdot\|_v}(Y; s) = -\int_{X_v^{\mathrm{an}}} \log \|s\|_v \, \delta_Y.$$

We now study the behavior of these objects with respect to adelic field extensions. Let $(\mathbb{F}, \mathfrak{N})$ be an extension of the adelic field $(\mathbb{K}, \mathfrak{M})$ as in Definition 2.3.5, and fix a place $w \in \mathfrak{N}_v$, so that $\mathbb{F}_w$ is a finite extension of the local field $\mathbb{K}_v$. Let q be a closed point of X_v and consider the subscheme q_w of $X_w = X \times \mathrm{Spec}(\mathbb{F}_w)$ obtained by base change. Decompose

$$\mathrm{K}(q) \otimes_{\mathbb{K}_v} \mathbb{F}_w = \bigoplus_{j \in I} G_j$$

as a finite sum of local Artinian $\mathbb{F}_w$-algebras and, for each $j \in I$, denote by q_j the corresponding closed point of X_w. Then

$$[q_w] = \sum_{j \in I} l_{G_j}(G_j)\, q_j \quad \text{ and } \quad \delta_{[q_w]} = \sum_{j \in I} \dim_{\mathbb{F}_w}(G_j)\, \delta_{q_j^{\mathrm{an}}}$$

denote respectively the 0-cycle of X_w associated to q_w, and the Dirac measure supported on it.

The inclusion $\mathbb{K}_v \hookrightarrow \mathbb{F}_w$ induces a map of the corresponding analytic spaces

$$\pi \colon X_w^{\mathrm{an}} \longrightarrow X_v^{\mathrm{an}}. \tag{2.3.6}$$

In the non-Archimedean case, this map of Berkovich spaces is defined locally by restricting seminorms.

The following proposition gives the behavior of the measure associated to a 0-cycle with respect to field extensions.

Proposition 2.3.11. *With notation as above, let Y be a 0-cycle of X_v and set Y_w for the 0-cycle of X_w obtained by base extension. Then*

$$\pi_* \, \delta_{Y_w} = \delta_Y.$$

Proof. By the compatibility of the map π with restriction to subschemes, it follows that $\pi(q_j^{\mathrm{an}}) = q^{\mathrm{an}}$ for all $j \in I$. It follows that

$$\pi_* \, \delta_{[q_w]} = \sum_{j \in I} \dim_{\mathbb{F}_w}(G_j) \, \pi_* \, \delta_{q_j^{\mathrm{an}}} = \Big(\sum_{j \in I} \dim_{\mathbb{F}_w}(G_j) \Big) \delta_{q^{\mathrm{an}}} = [\mathrm{K}(q) : \mathbb{K}_v] \, \delta_{q^{\mathrm{an}}} = \delta_q.$$

□

Let D be a Cartier divisor on X and $\| \cdot \|_v$ a metric on $\mathcal{O}(D)_v^{\mathrm{an}}$. The extension of this metric to a metric $\| \cdot \|_w$ on the analytic line bundle $\mathcal{O}(D)_w^{\mathrm{an}}$ on X_w^{an} is obtained by taking the inverse image with respect to the map π in (2.3.6), that is

$$\| \cdot \|_w = \pi^* \, \| \cdot \|_v. \tag{2.3.7}$$

Proposition 2.3.11 implies directly the invariance of the local height with respect to adelic field extensions.

Proposition 2.3.12. *With notation as above, let Y be a 0-cycle of X_v and s a rational section of $\mathcal{O}(D)_v^{\mathrm{an}}$ that is regular and non-vanishing on the support of Y. Set Y_w and $s_w = \pi^* s$ for the 0-cycle and rational section obtained by base extension. Then*

$$\mathrm{h}_{\| \cdot \|_w}(Y_w, s_w) = \mathrm{h}_{\| \cdot \|_v}(Y, s).$$

To define global heights of cycles over an adelic field, we consider adelic families of metrics on the Cartier divisor D satisfying a certain compatibility condition.

Definition 2.3.13. An *(adelic) metric* on D is a collection $\| \cdot \|_v$ of metrics on $\mathcal{O}(D)_v^{\mathrm{an}}$, for $v \in \mathfrak{M}$, such that, for every point $p \in X(\overline{\mathbb{K}})$ and a choice of a rational section s of $\mathcal{O}(D)$ that is regular and non-vanishing at p and of an adelic field extension $(\mathbb{F}, \mathfrak{N})$ such that $p \in X(\mathbb{F})$,

$$\| s(p_w^{\mathrm{an}}) \|_w = 1 \tag{2.3.8}$$

for all but a finite number of $w \in \mathfrak{N}$. We denote by $\overline{D} = (D, (\| \cdot \|_v)_{v \in \mathfrak{M}})$ the corresponding *(adelically) metrized divisor* on X.

In addition, $\overline{D}$ is *semipositive* if each of its v-adic metrics is semipositive in the sense of [19, Definition 1.4.1].

The condition (2.3.8) does not depend on the choice of the rational section s and of the adelic field extension $(\mathbb{F}, \mathfrak{N})$.

Remark. *When $\mathbb{K}$ is a global field, the classical notion of compatibility for a collection of metrics $\|\cdot\|_v$ on $\mathcal{O}(D)_v^{\mathrm{an}}$, $v \in \mathfrak{M}$, is that of being quasi-algebraic, in the sense that there is an integral model that induces all but a finite number of these metrics [19, Definition 1.5.13].*

By Proposition 1.5.14 in loc. cit.*, a quasi-algebraic metrized divisor $\overline{D}$ is adelic in the sense of Definition 2.3.13. The converse is not true, as it is easy to construct toric adelic metrized divisors that are not quasi-algebraic (Remark 2.3.3).*

For a 0-cycle Y of X and a place $v \in \mathfrak{M}$, we denote by Y_v the 0-cycle of X_v defined by base change. When $Y = p$ is a closed point of X, by Lemma 2.3.4 applied to the finite extension $\mathrm{K}(p)$ of $\mathbb{K}$, the 0-dimensional subscheme $p_v = p \times \mathrm{Spec}(\mathbb{K}_v)$ of X_v decomposes as

$$p_v = \mathrm{Spec}(\mathrm{K}(p) \otimes_{\mathbb{K}} \mathbb{K}_v) \simeq \coprod_{w \in \mathfrak{N}_v} \mathrm{Spec}(E_w),$$

where the E_i's are the Artinian $\mathbb{K}_v$-algebras in (2.3.3). Let q_w, $w \in \mathfrak{N}_v$, be the irreducible components of this subscheme. Then, the associated 0-cycle of X_v writes down as

$$[p_v] = \sum_{w \in \mathfrak{N}_v} l_{E_w}(E_w)\, q_w$$

and, for each $w \in \mathfrak{N}_v$, we have $\mathrm{K}(q_w) \simeq \mathrm{K}(p)_w$. For an arbitrary Y, the 0-cycle Y_v is obtained by linearity.

Let $\overline{D} = (D, (\|\cdot\|_v)_{v \in \mathfrak{M}})$ be a metrized divisor on X, Y a 0-cycle of X and s a rational section of $\mathcal{O}(D)$ that is is regular and non-vanishing on the support of Y. For each place $v \in \mathfrak{M}$, we set

$$\mathrm{h}_{\overline{D},v}(Y; s) = \mathrm{h}_{\|\cdot\|_v}(Y_v; s),$$

where Y_v is the 0-cycle of X_v obtained by base change. The condition that $\overline{D}$ is adelic implies that $\mathrm{h}_{\overline{D},v}(Y; s) = 0$ for all but a finite number of places.

If s' is another rational section of $\mathcal{O}(D)$ that is regular and non-vanishing on $|Y|$, then $s' = fs$ with $f \in \mathrm{K}(X)^\times$ and, for $v \in \mathfrak{M}$,

$$\mathrm{h}_{\overline{D},v}(Y; s') = \mathrm{h}_{\overline{D},v}(Y; s) - \log|\gamma|_v \tag{2.3.9}$$

where $Y = \sum_p \mu_p\, p$ and $\gamma = \prod_p f(p)^{\mu_p} \in \mathbb{K}^\times$.

Definition 2.3.14. Let $\overline{D}$ be a metrized divisor on X and Y a 0-cycle of X. The *global height* of Y with respect to $\overline{D}$ is defined as

$$\mathrm{h}_{\overline{D}}(Y) = \sum_{v \in \mathfrak{M}} n_v\, \mathrm{h}_{\overline{D},v}(Y; s), \tag{2.3.10}$$

with s a rational section of $\mathcal{O}(D)$ that is is regular and non-vanishing on $|Y|$.

The local heights in (2.3.10) are zero for all but a finite number of places, and so this sum is finite. The equality (2.3.9) together with the product formula imply that this sum does not depend on the rational section s.

Given a metrized divisor $\overline{D}$ on X and an adelic field extension $(\mathbb{F},\mathfrak{N})$, we denote by $\overline{D}_{\mathbb{F}}$ the metrized divisor on $X_{\mathbb{F}}$ obtained by extending the v-adic metrics of $\overline{D}$ as in (2.3.7).

Proposition 2.3.15. *Let $\overline{D}$ be a metrized divisor on X, Y a 0-cycle of X and $(\mathbb{F},\mathfrak{N})$ an adelic field extension of $(\mathbb{K},\mathfrak{M})$. Then*

$$\mathrm{h}_{\overline{D}_{\mathbb{F}}}(Y_{\mathbb{F}}) = \mathrm{h}_{\overline{D}}(Y).$$

Proof. Let s be a rational section of $\mathcal{O}(D)$ that is is regular and non-vanishing on $|Y|$ and $v\in\mathfrak{M}$. By Propositions 2.3.12 and 2.3.6(1),

$$\sum_{w\in\mathfrak{N}_v} n_w\ \mathrm{h}_{\overline{D}_{\mathbb{F}},w}(Y_{\mathbb{F}},s) = \sum_{w\in\mathfrak{N}_v} n_w\ \mathrm{h}_{\overline{D},v}(Y,s) = n_v\ \mathrm{h}_{\overline{D},v}(Y,s).$$

The statement follows by summing over all the places of $\mathbb{K}$. □

Since the global height is invariant under field extension, it induces a notion of global height for algebraic points, that is, a well-defined function

$$\mathrm{h}_{\overline{D}}\colon X(\overline{\mathbb{K}}) \longrightarrow \mathbb{R}.$$

When $\mathbb{K}$ is a global field, this notion coincides with the one in [18, Definition 2.2].

Now we turn to cycles of arbitrary dimension. Let V be a k-dimensional irreducible subvariety of X and $\overline{D}_0,\dots,\overline{D}_{k-1}$ a family of k semipositive metrized divisors on X. For each place $v\in\mathfrak{M}$, we can associate to this data a measure on X_v^{an} denoted by

$$\mathrm{c}_1(\overline{D}_0)\wedge\cdots\wedge\mathrm{c}_1(\overline{D}_{k-1})\wedge\delta_{V_v^{\mathrm{an}}}$$

and called the *v-adic Monge-Ampère measure* of V and $\overline{D}_0,\dots,\overline{D}_{k-1}$ [19, Definition 1.4.6] and [22, Définition 2.4]. For a k-cycle Y of X, this notion extends by linearity to a signed measure on X_v^{an}, denoted by $\mathrm{c}_1(\overline{D}_0)\wedge\cdots\wedge\mathrm{c}_1(\overline{D}_{k-1})\wedge\delta_{Y_v^{\mathrm{an}}}$. It is supported on $|Y_v|^{\mathrm{an}}$ and has total mass equal to the degree $\deg_{D_0,\dots,D_{k-1}}(Y)$.

We recall the notion of local height of cycles from [19, Definition 1.4.11].

Definition 2.3.16. Let Y be a k-cycle of X and, for $i=0,\dots,k$, let $(\overline{D}_i,s_i)$ be a semipositive metrized divisor on X and a rational section of $\mathcal{O}(D_i)$ such that $\mathrm{div}(s_0),\dots,\mathrm{div}(s_k)$ intersect Y properly (Definition 2.2.1). For $v\in\mathfrak{M}$, the *local height* of Y with respect to $(\overline{D}_0,s_0),\dots,(\overline{D}_k,s_k)$ is inductively defined by the rule

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y;s_0,\dots,s_k) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1},v}(\mathrm{div}(s_k)\cdot Y;s_0,\dots,s_{k-1})$$

$$-\int_{X_v^{\mathrm{an}}} \log ||s_k||_{k,v} \mathrm{c}_1(\overline{D}_0) \wedge \ldots \wedge \mathrm{c}_1(\overline{D}_{k-1}) \wedge \delta_{Y_v^{\mathrm{an}}}$$

and the convention that the local height of the cycle $0 \in Z_{-1}(X)$ is zero.

Remark.

1. *The local height is linear with respect to the group structure of* $Z_k(X)$. *In particular, the local heights of the cycle* $0 \in Z_k(X)$ *are zero.*
2. *For a closed point of* X *and* $v \in \mathfrak{M}$, *it the* v*-adic Monge-Ampère measure coincides with the weighted Dirac measure in* (2.3.5), *see for instance [19, page 17 and Definition 1.3.15]. Hence, for* 0*-cycles, the local heights in Definitions 2.3.10 and 2.3.16 coincide.*

The following notion is the arithmetic analogue of global sections of a line bundle, and Proposition 2.3.18 below is an analogue for local heights of Proposition 2.2.5.

Definition 2.3.17. Let $\overline{D} = (D, (||\cdot||_v)_{v\in\mathfrak{M}})$ be a metrized divisor on X. A global section s of $\mathcal{O}(D)$ is $\overline{D}$*-small* if, for all $v \in \mathfrak{M}$,

$$\sup_{q\in X_v^{\mathrm{an}}} \|s(q)\|_v \le 1.$$

Proposition 2.3.18. *Let* Y *be an effective* k*-cycle of* X *and, for* $i = 0, \ldots, k$, *let* $(\overline{D}_i, s_i)$ *be a semipositive metrized divisor on* X *and a rational section of* $\mathcal{O}(D_i)$ *such that* $\mathrm{div}(s_0), \ldots, \mathrm{div}(s_k)$ *intersect* Y *properly and such that* s_k *is* $\overline{D}_k$*-small. Then, for each place* $v \in \mathfrak{M}$,

$$\mathrm{h}_{\overline{D}_0,\ldots,\overline{D}_{k-1},v}(\mathrm{div}(s_k)\cdot Y; s_0, \ldots, s_{k-1}) \le \mathrm{h}_{\overline{D}_0,\ldots,\overline{D}_k,v}(Y; s_0, \ldots, s_k).$$

Proof. Since the cycle Y is effective and the metrized divisors $\overline{D}_i$ are semipositive, their v-adic Monge-Ampere measure is a measure, that is, it takes only nonnegative values. Since the global section s_k is $\overline{D}_k$-small, $\log ||s_k(q)||_{k,v} \le 0$ for all $q \in X_v^{\mathrm{an}}$. The inequality follows then from the inductive definition of the local height. □

Our next step is to define global heights for cycles over an adelic field. We first state an auxiliary result specifying the behavior of local heights with respect to change of sections, extending (2.3.9) to the higher dimensional case. The following lemma and its proof are similar to [39, Corollary 3.8].

Lemma 2.3.19. *Let* Y *be a* k*-cycle of* X *and* $\overline{D}_0, \ldots, \overline{D}_k$ *semipositive metrized divisors on* X. *Let* s_i, s_i' *be rational sections of* $\mathcal{O}(D_i)$, $i = 0, \ldots, k$, *such that both* $\mathrm{div}(s_0), \ldots, \mathrm{div}(s_k)$ *and* $\mathrm{div}(s_0'), \ldots, \mathrm{div}(s_k')$ *intersect* Y *properly. Then there exists* $\gamma \in \mathbb{K}^\times$ *such that, for all* $v \in \mathfrak{M}$,

$$\mathrm{h}_{\overline{D}_0,\ldots,\overline{D}_k,v}(Y; s_0', \ldots, s_k') = \mathrm{h}_{\overline{D}_0,\ldots,\overline{D}_k,v}(Y; s_0, \ldots, s_k) - \log|\gamma|_v. \tag{2.3.11}$$

Proof. Let s''_i be a rational section of $\mathcal{O}(D_i)$, $i=0,\dots,k$, such that the $(s''_0,\dots,s''_k)$ is generic.

Notice that, by the genericity of $(s''_0,\dots,s''_k)$, for any choice of $r=0,\dots,k$ and any permutation $i_0,\dots,i_k$ of $0,\dots,k$, we have that

$$\operatorname{div}(s_{i_0}),\dots,\operatorname{div}(s_{i_r}),\operatorname{div}(s''_{i_{r+1}}),\dots,\operatorname{div}(s''_{i_k})$$

intersect Y properly. We proceed by proving (2.3.11) with the s''_i's in the place of the s'_i's. That is, there exists a $\tilde{\gamma}\in\mathbb{K}^\times$ such that, for all $v\in\mathfrak{M}$,

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y;s''_0,\dots,s''_k)=\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y;s_0,\dots,s_k)-\log|\tilde{\gamma}|_v. \tag{2.3.12}$$

Consider the particular case when $s_i=s''_i$ for $i=0,\dots,k-1$. Set $s''_k=fs_k$ with $f\in\mathrm{K}(X)^\times$, and $\big(\prod_{i=0}^{k-1}\operatorname{div}(s_i)\big)\cdot Y=\sum_p\mu_p\,p$. By [19, Theorem 1.4.17(3)], the equality (2.3.12) holds with $\tilde{\gamma}_k\in\mathbb{K}^\times$ given by

$$\tilde{\gamma}_k=\prod_p f(p)^{\mu_p}.$$

By [19, Theorem 1.4.17(1)], the local height is symmetric in the pairs $(\overline{D_i},s_i)$. Hence, we can reorder the metrized line bundles and sections, and iterate the above construction for every $i=0,\dots,k$. The resulting $\tilde{\gamma}$ in (2.3.12) is obtained by multiplying each of the $\tilde{\gamma}_i$'s.

Analogously, we can proof (2.3.12) replacing the s_i's by the s'_i's. By combining both these equalities, we obtain (2.3.11). □

We consider the following notions of positivity of metrized divisors.

Definition 2.3.20. Let $\overline{D}$ be a metrized divisor on X.

1. $\overline{D}$ is *nef* if D is nef, $\overline{D}$ is semipositive, and $\mathrm{h}_{\overline{D}}(p)\geq 0$ for every closed point p of X.
2. $\overline{D}$ is *generated by small sections* if, for every closed point $p\in X$, there is a $\overline{D}$-small section s such that $p\notin|\operatorname{div}(s)|$.

Lemma 2.3.21. *Let Y be an effective k-cycle of X and $(\overline{D}_i,s_i)$ semipositive metrized divisors on X together with a rational section of $\mathcal{O}(D_i)$, $i=0,\dots,k$, such that the divisors $\operatorname{div}(s_0),\dots,\operatorname{div}(s_k)$ intersect Y properly. Suppose that $\overline{D}_i$, $i=1,\dots,k$, are generated by small sections. Then there exists $\zeta\in\mathbb{K}^\times$ such that, for all $v\in\mathfrak{M}$,*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y;s_0,\dots,s_k)\geq\log|\zeta|_v+\mathrm{h}_{\overline{D}_0,v}\bigg(\bigg(\prod_{i=1}^k\operatorname{div}(s_i)\bigg)\cdot Y,s_0\bigg).$$

Proof. For $k=0$, the statement is obvious, so we only consider the case when $k\geq1$. By Lemma 2.3.19, it is enough to prove the statement for any particular choice of rational sections s_i, provided that their associated Cartier divisors intersect Y properly.

We can also reduce without loss of generality to the case when $Y = V$ is an irreducible variety of dimension k. We can then choose rational sections s_i, $i = 0, \dots, k$, such that each s_i is $\overline{D}_i$-small. By Proposition 2.3.18,

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(V; s_0, \dots, s_k) \geq \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1},v}(\mathrm{div}(s_k) \cdot V; s_0, \dots, s_{k-1}).$$

Since $\mathrm{div}(s_k) \cdot V$ is an effective $(k-1)$-cycle, the statement follows by induction on k. $\square$

Proposition-Definition 2.3.22. *Let Y be an effective k-cycle of X, and $\overline{D}_0, \dots, \overline{D}_k$ semipositive metrized divisors on X such that $\overline{D}_1, \dots, \overline{D}_k$ are generated by small sections. Let s_i be a rational section of $\mathcal{O}(D_i)$, $i = 0, \dots, k$, such that* $\mathrm{div}(s_0), \dots, \mathrm{div}(s_k)$ *intersect Y properly. The* global height *of Y with respect to $\overline{D}_0, \dots, \overline{D_k}$ is defined as the sum*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k}(Y) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k). \tag{2.3.13}$$

This sum converges to an element in $\mathbb{R} \cup \{+\infty\}$, and its value does not depend on the choice of the s_i's.

Proof. The existence of rational sections s_i such that $\mathrm{div}(s_0), \dots, \mathrm{div}(s_k)$ intersects Y properly follows from the moving lemma, with the hypothesis that X is projective.

By Lemma 2.3.21 and the fact that the local heights of 0-cycles are zero for all but a finite number of places, the local heights in (2.3.13) are non negative, except for a finite number of v's. Hence, the sum converges to an element in $\mathbb{R} \cup \{+\infty\}$. Lemma 2.3.19 and the product formula imply that the value of this sum does not depend on the choice of the s_i's. $\square$

Remark. *This definition generalizes the notion of global height of cycles of varieties over global fields in [19, §1.5], to cycles of varieties over an arbitrary adelic field, in the case when the considered metrized divisors are generated by small sections.*

In the context of varieties over global fields, the local heights of a given cycle are zero for all but a finite number of places [19, Proposition 1.5.14], and so their global height is a real number given as a weighted sum of a finite number local heights. In our present generality, the sum in (2.3.13) *might contain an infinite number of nonzero terms. We will see that, in the toric situation, these global heights are nonnegative real numbers, different from $+\infty$.*

The following results are arithmetic analogues of Proposition 2.2.5 and Corollary 2.2.6.

Proposition 2.3.23. *Let Y be an effective k-cycle of X, and $\overline{D}_0, \dots, \overline{D}_k$ semipositive metrized divisors on X such that $\overline{D}_0$ is nef and $\overline{D}_1, \dots, \overline{D}_k$ are generated by small sections. Let s_k be a $\overline{D}_k$-small section. Then*

$$0 \leq \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1}}(\mathrm{div}(s_k) \cdot Y) \leq \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k}(Y).$$

Proof. We reduce without loss of generality to the case when $Y = V$ is an irreducible subvariety of dimension k. If $V \subset |\operatorname{div}(s_k)|$, the first inequality is clear. For the second inequality, we choose rational sections s_i, $i = 0, \dots, k-1$, and s'_k such that $\operatorname{div}(s_0), \dots, \operatorname{div}(s_{k-1}), \operatorname{div}(s'_k)$ intersect Y properly. Using Lemmas 2.3.19 and 2.3.21, the product formula and the fact that $\overline{D}_0$ is nef, we deduce that $\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k}(Y) \geq 0$.

Otherwise, $V \not\subset |\operatorname{div}(s_k)|$ and we choose rational sections s_i, $i = 0, \dots, k-1$, such that $\operatorname{div}(s_0), \dots, \operatorname{div}(s_k)$ intersect Y properly. The first inequality follows by applying the argument above to $\operatorname{div}(s_k) \cdot Y$, whereas the second one is given by Proposition 2.3.18. □

Corollary 2.3.24. *Let $\overline{D}_0, \dots, \overline{D}_n$ be semipositive metrized divisors on X such that $\overline{D}_0$ is nef and $\overline{D}_1, \dots, \overline{D}_n$ are generated by small sections. Let s_i be a $\overline{D}_i$-small section, $i = 1, \dots, n$. Then*

$$0 \leq \mathrm{h}_{\overline{D}_0}\left(\prod_{i=1}^{n} \operatorname{div}(s_i)\right) \leq \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_n}(X).$$

2.3.3 Metrics and heights on toric varieties

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula. Let $M \simeq \mathbb{Z}^n$ be a lattice and $\mathbb{T} \simeq \mathbb{G}^n_{\mathrm{m},\mathbb{K}}$ its associated torus over $\mathbb{K}$ as in (2.2.4). For $v \in \mathfrak{M}$, we denote by $\mathbb{T}^{\mathrm{an}}_v$ the v-adic analytification of $\mathbb{T}$, and by $\mathbb{S}_v$ its maximal compact subgroup. In the Archimedean case, $\mathbb{S}_v$ is homeomorphic to the polycircle $(S^1)^n$, whereas in the non-Archimedean case, it is a compact analytic group, see [19, §4.2] for a description. Moreover, there is a map defined, in a given splitting, as

$$\begin{aligned} \mathrm{val}_v \colon \mathbb{T}^{\mathrm{an}}_v &\longrightarrow N_{\mathbb{R}} \\ (x_1, \dots, x_n) &\longmapsto (-\log|x_1|_v, \dots, -\log|x_n|_v). \end{aligned}$$

This map does not depend on the choice of the splitting, and $\mathbb{S}_v$ coincides with its fiber over the point $0 \in N_{\mathbb{R}}$.

Let X be a projective toric variety with torus $\mathbb{T}$ given by a regular complete fan Σ on $N_{\mathbb{R}}$, and D a toric Cartier divisor on X given by a virtual support function Ψ_D on Σ. Recall that X contains $\mathbb{T}$ as a dense open subset. Let $||\cdot||_v$ be a toric v-adic metric on D, that is, a metric on the analytic line bundle $\mathcal{O}(D)^{\mathrm{an}}_v$ that is invariant under the action of $\mathbb{S}_v$. This allows to define a continuous function $\psi_{||\cdot||_v} \colon N_{\mathbb{R}} \to \mathbb{R}$, called *$v$-adic metric function* associated to $||\cdot||_v$, given by

$$\psi_{||\cdot||_v}(u) = \log ||s_D(p)||_v, \tag{2.3.14}$$

for any $p \in \mathbb{T}^{\mathrm{an}}_v$ with $\mathrm{val}_v(p) = u$ and where s_D is the distinguished rational section of $\mathcal{O}(D)$. This function satisfies that $|\psi_{||\cdot||_v} - \Psi_D|$ is bounded on $N_{\mathbb{R}}$ and moreover, this difference extends to a continuous function on N_Σ, the compactification of $N_\mathbb{R}$ induced by the fan Σ. Indeed, the assignment

$$||\cdot||_v \longmapsto \psi_{||\cdot||_v} \tag{2.3.15}$$

is a one-to-one correspondence between the set of toric v-adic metrics on D and the set of such continuous functions on $N_\mathbb{R}$ [19, Proposition 4.3.10]. In particular, the toric v-adic metric on D associated to the virtual support function Ψ_D is called the *canonical* v-adic toric metric of D and is denoted by $||\cdot||_{v,\mathrm{can}}$.

Furthermore, when $\|\cdot\|_v$ is semipositive, $\psi_{\|\cdot\|_v}$ is a concave function and it verifies that $|\psi_{||\cdot||_v} - \Psi_D|$ is bounded on $N_\mathbb{R}$, and the assignment in (2.3.15) gives a one-to-one correspondence between the set of semipositive toric v-adic metrics on D and the set of such concave functions on $N_\mathbb{R}$.

When $\|\cdot\|_v$ is semipositive, we also consider a continuous concave function on the polytope $\vartheta_{\|\cdot\|_v} \colon \Delta_D \to \mathbb{R}$ defined as the Legendre-Fenchel dual of $\psi_{\|\cdot\|_v}$, that is

$$\vartheta_{\|\cdot\|_v}(x) = \inf_{u \in N_\mathbb{R}} \langle x, u\rangle - \psi_{\|\cdot\|_v}(u).$$

We call this function, the *v-adic roof function* associated to $||\cdot||_v$. The assignment $\|\cdot\|_v \mapsto \vartheta_{\|\cdot\|_v}$ is a one-to-one correspondence between the set of semipositive toric v-adic metrics on D and that of continuous concave functions on Δ_D. Under this assignment, the canonical v-adic toric metric of D corresponds to the zero function on Δ_D.

Definition 2.3.25. An *(adelic) toric metric* on D is a collection of toric v-adic metrics $(\|\cdot\|_v)_{v\in\mathfrak{M}}$, such that $\|\cdot\|_v = ||\cdot||_{v,\mathrm{can}}$ for all but a finite number of $v \in \mathfrak{M}$. We denote by $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ the corresponding *(adelic) toric metrized divisor* on X.

Example 2.3.26. The collection $(\|\cdot\|_{v,\mathrm{can}})_{v\in\mathfrak{M}}$ of v-adic toric metrics on D is adelic in the sense of Definition 2.3.25. We denote by $\overline{D}^{\mathrm{can}}$ the corresponding *canonical* toric metrized divisor on X.

Let $\overline{D}$ be a toric metrized divisor on X. For each $v \in \mathfrak{M}$, we set

$$\psi_{\overline{D},v} = \psi_{\|\cdot\|_v} \quad \text{ and } \quad \vartheta_{\overline{D},v} = \vartheta_{\|\cdot\|_v}$$

for the associated *v-adic metric function* and *v-adic roof function*, respectively.

Proposition 2.3.27. *Let $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ be toric divisor together with a collection of toric v-adic metrics. If $\overline{D}$ is adelic in the sense of Definition 2.3.25, then it is also adelic in the sense of Definition 2.3.13. Moreover, both definitions coincide in the semipositive case.*

Proof. Let $p \in X(\overline{\mathbb{K}})$ and choose an adelic field extension $(\mathbb{F}, \mathfrak{N})$ such that $p \in X(\mathbb{F})$. Then $p_{\mathbb{F}}$ is a rational point of $X_{\mathbb{F}}$ and the inclusion

$$\iota\colon p_{\mathbb{F}} \longrightarrow X_{\mathbb{F}}$$

is an equivariant map. Hence the inverse image $\iota^*\overline{D}$ is an adelic toric metric on $p_{\mathbb{F}}$ and so, for $w \in \mathfrak{N}$,

$$\log \|p_{\mathbb{F}}\|_w = \psi_{\iota^*\overline{D},w}(0),$$

and this quantity vanishes for all but the finite number of $w \in \mathfrak{N}$ such that $\|\cdot\|_w$ is not the canonical metric. Since this holds for all $p \in X(\overline{\mathbb{K}})$, we conclude that $\overline{D}$ is adelic in the sense of Definition 2.3.13.

For the second statement, assume that $\overline{D}$ is semipositive and adelic in the sense of Definition 2.3.13. Let $x_i \in M$, $i = 1, \dots, s$, be the vertices of the lattice polytope Δ_D. By [19, Example 2.5.13], there is an n-dimensional cone $\sigma_i \in \Sigma$ corresponding to x_i under the Legendre-Fenchel correspondence, $i = 1, \dots, s$. Each of these n-dimensional cones corresponds to a 0-dimensional orbit p_i of X. Denote by $\iota_i\colon p_i \hookrightarrow X$ the inclusion of this orbit.

Fix $1 \leq i \leq s$. Modulo a translation, we can assume without loss of generality that $x_i = 0$. By [19, Proposition 4.8.9], for $v \in \mathfrak{M}$,

$$\vartheta_{\overline{D},v}(x_i) = \vartheta_{\iota_i^*\overline{D},v}(0) = -\log \|s_D(p_i)\|_v.$$

Hence $\vartheta_{\overline{D},v}(x_i) = 0$ for all but a finite number of v's.

On the other hand, let x_0 be the distinguished point of X, which coincides with the neutral element of $\mathbb{T}$, and denote by $\iota_0\colon x_0 \hookrightarrow X$ its inclusion. By [19, Proposition 4.8.10],

$$\max_{x \in \Delta_D} \vartheta_{\overline{D},v}(x) = \vartheta_{\iota_0^*\overline{D},v}(0) = -\log \|s_D(x_0)\|_v.$$

Hence $\max_{x\in\Delta_D} \vartheta_{\overline{D},v}(x) = 0$ for all but a finite number of v's.

For all $v \in \mathfrak{M}$ such that $\vartheta_{\overline{D},v}(x_i) = 0$ for all i and $\max_{x\in\Delta_D} \vartheta_{\overline{D},v}(x) = 0$, we have that $\vartheta_{\overline{D},v} \equiv 0$ because this local roof function is a concave function on Δ_D. Hence, $\|\cdot\|_v$ coincides with the v-adic canonical metric of D for all these places. □

Remark. *In the general non-semipositive case, Definitions 2.3.25 and 2.3.13 do not coincide. For instance, when $X = \mathbb{P}^1_{\mathbb{K}}$, a collection of metrics $\|\cdot\|_v$, $v \in \mathfrak{M}$, satisfies Definition 2.3.13 if and only if its associated metric functions satisfy that*

$$\psi_{\overline{D},v}(0) = 0 \quad \text{and} \quad \lim_{u\to\pm\infty} \psi_{\overline{D},v}(u) - \Psi_D(u) = 0$$

for all but a finite number of places. In the absence of convexity, these conditions do not imply that $\psi_{\overline{D},v} = \Psi_D$ for all but a finite number of places.

Proposition 2.3.28. *Let D be a toric Cartier divisor on X.*

1. *The assignment $\overline{D} \mapsto (\psi_{\overline{D},v})_{v\in\mathfrak{M}}$ is a one-to-one correspondence between the set of semipositive toric metrics on D, and the set of families of concave functions $(\psi_v)_{v\in\mathfrak{M}}$ on $N_{\mathbb{R}}$ such that $|\psi_v - \Psi_D|$ is bounded for all v, and $\psi_v = \Psi_D$ for all but a finite number of $v \in \mathfrak{M}$.*

2. *The assignment $\overline{D} \mapsto (\vartheta_{\overline{D},v})_{v\in\mathfrak{M}}$ is a one-to-one correspondence between the set of semipositive toric metrics on D and the set of families of continuous concave functions $(\vartheta_v)_{v\in\mathfrak{M}}$ on Δ_D such that $\vartheta_v = 0$ for all but a finite number of $v \in \mathfrak{M}$.*

A classical example of toric metrized divisors are those given by the inverse image of an equivariant map to a projective space equipped with the canonical metric on its universal line bundle. Below we describe this example and we refer to [19, Example 5.1.16] for the technical details.

Let $\boldsymbol{m} = (m_0, \dots, m_r) \in M^{r+1}$ and $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$, with $r \geq 0$. The monomial map associated to this data is defined as

$$\begin{aligned} \varphi_{\boldsymbol{m},\boldsymbol{\alpha}} \colon \mathbb{T} &\longrightarrow \mathbb{P}^r_{\mathbb{K}}, \\ p &\longmapsto \big(\alpha_0\chi^{m_0}(p) : \dots : \alpha_r\chi^{m_r}(p)\big). \end{aligned} \tag{2.3.16}$$

Let Σ be a regular fan in $N_{\mathbb{R}}$ compatible with the polytope $\Delta = \mathrm{conv}(m_0, \dots, m_r) \subset M_{\mathbb{R}}$, in the sense that the support function Ψ_Δ is a virtual support function on Σ. For a toric variety X with torus $\mathbb{T}$ corresponding to the fan Σ, the monomial map (2.3.16) extends to an equivariant map $X \to \mathbb{P}^r_{\mathbb{K}}$, also denoted by $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$.

Example 2.3.29. With notation as above, let $\overline{E}^{\mathrm{can}}$ be the divisor of the hyperplane at infinity of $\mathbb{P}^r_{\mathbb{K}}$, equipped with the canonical metric at all places. Then $D = \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}}E$ is the nef toric Cartier divisor on X corresponding to the translated polytope $\Delta - m_0$. We consider the semipositive toric metrized divisor $\overline{D} = \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}}\overline{E}$ on X.

For each $v \in \mathfrak{M}$, the v-adic metric function of $\overline{D}$, $\psi_{\overline{D},v} \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}$, is given by

$$\psi_{\overline{D},v}(u) = \min_{0\leq j\leq r}\Big(\langle m_j - m_0, u\rangle - \log\Big|\frac{\alpha_j}{\alpha_0}\Big|_v\Big).$$

The polytope corresponding to D is $\Delta - m_0$ and, for each $v \in \mathfrak{M}$, the v-adic roof function of $\overline{D}$ is given by

$$\vartheta_{\overline{D},v}(x) = \max_{\boldsymbol{\lambda}} \sum_{j=0}^{r} \lambda_j \log|\alpha_j|_v - \log|\alpha_0|_v,$$

the maximum being over the vectors $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_r) \in \mathbb{R}^{r+1}_{\geq 0}$ with $\sum_{j=0}^r \lambda_j = 1$ such that $\sum_{j=0}^r \lambda_j(m_j - m_0) = x$. In other words, this the piecewise affine concave function on $\Delta - m_0$ parametrizing the upper envelope of the extended polytope

$$\mathrm{conv}\left((m_j - m_0, \log|\alpha_j/\alpha_0|_v)_{0\leq j\leq r}\right) \subset M_{\mathbb{R}} \times \mathbb{R}.$$

Definition 2.3.30. For $i = 0, \dots, n$, let $g_i : \Delta_i \to \mathbb{R}$ be a concave function on a convex body $\Delta_i \subset M_\mathbb{R}$. The *mixed integral* of $g_0, \dots, g_n$ is defined as

$$\mathrm{MI}_M(g_0, \dots, g_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0 \le i_0 < \dots < i_j \le n} \int_{\Delta_{i_0} + \dots + \Delta_{i_j}} g_{i_0} \boxplus \dots \boxplus g_{i_j} \, \mathrm{d}\,\mathrm{vol}_M,$$

where $\Delta_{i_0} + \dots + \Delta_{i_j}$ denotes the Minkowski sum of polytopes, and $g_{i_0} \boxplus \dots \boxplus g_{i_j}$ the sup-convolution of concave function, which is the function on $\Delta_{i_0} + \dots + \Delta_{i_j}$ defined as

$$g_{i_0} \boxplus \dots \boxplus g_{i_j}(x) = \sup \big(g_{i_0}(x_{i_0}) + \dots + g_{i_j}(x_{i_j}) \big),$$

where the supremum is taken over $x_{i_l} \in \Delta_{i_l}$, $l = 0, \dots, j$, such that $x_{i_0} + \dots + x_{i_j} = x$.

The mixed integral is symmetric and additive in each variable with respect to the sup-convolution. Moreover, for a concave function $g \colon \Delta \to \mathbb{R}$ on a convex body Δ, we have $\mathrm{MI}_M(g, \dots, g) = (n+1)! \int_\Delta g \, \mathrm{d}\,\mathrm{vol}_M$, see [67, §8] for details.

The following is a restricted version of a result by Burgos Gil, Philippon and Sombra, giving the global height of a toric variety with respect to a family of semipositive toric metrized divisors in terms of the mixed integrals of the associated local roof functions [19, Theorem 5.2.5].

Theorem 2.3.31. *Let $\overline{D}_i$, $i = 0, \dots, n$, be semipositive toric metrized divisors on X such that $\overline{D}_1, \dots, \overline{D}_n$ are generated by small sections. Then*

$$\mathrm{h}_{\overline{D}_0, \dots, \overline{D}_n}(X) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0, v}, \dots, \vartheta_{\overline{D}_n, v}). \tag{2.3.17}$$

Remark. *The result in [19, Theorem 5.2.5] is more general. Given semipositive toric metrized divisors $\overline{D}_i$, $i = 0, \dots, n$, and rational sections s_i such that* $\mathrm{div}(s_0), \dots, \mathrm{div}(s_n)$ *intersect X properly, the corresponding local heights are zero except for a finite number of places, and the formula* (2.3.17) *holds without any extra positivity assumption.*

2.4 Arithmetic Bernštein-Kušnirenko

In this section we first prove the main results of this chapter, Theorem 2.4.5 and Corollary 2.4.8, which give bounds on the height of 0-cycles coming from systems of Laurent polynomials. Furthermore, we apply these results to more concrete settings: we present two families of examples and compare the actual height of the the 0-cycles with the bounds provided by our results. Finally, we give an application bounding the height of the resultant of a 0-cycle defined by a system of Laurent polynomials.

2.4.1 Main theorem

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula. Let $f \in \mathbb{K}[M]$ be a Laurent polynomial and $\Delta \subset M_{\mathbb{R}}$ its Newton polytope. Let X be a projective toric variety over $\mathbb{K}$ given by a fan on $N_{\mathbb{R}}$ that is compatible with Δ, and D the Cartier divisor on X given by this polytope. To prove our main theorem, we first construct a toric metric on D such that the associated toric metrized divisor $\overline{D}$ is semipositive and generated by small sections, and the global section of $\mathcal{O}(D)$ associated to f is $\overline{D}$-small. We obtain this metrized divisor as the inverse image of a metrized divisor on a projective space.

For $r \geq 0$, let $\mathbb{P}^r_{\mathbb{K}}$ be the r-dimensional projective space over $\mathbb{K}$ and E the divisor of the hyperplane at infinity. We denote by $\overline{E}$ this Cartier divisor equipped with the ℓ^1-norm at the Archimedean places, and the canonical one at the non-Archimedean ones. This metric is defined, for $p = (p_0 : \cdots : p_s) \in \mathbb{P}^s_{\mathbb{K}}(\overline{\mathbb{K}}_v)$ and a global section s of $\mathcal{O}(E)$ corresponding to a linear form $\rho_s \in \mathbb{K}[x_0, \dots, x_s]$, by

$$\|s(p)\|_v = \begin{cases} \dfrac{|\rho_s(p_0, \dots, p_s)|_v}{\sum_j |p_j|_v} & \text{if } v \text{ is Archimedean,} \\ \dfrac{|\rho_s(p_0, \dots, p_s)|_v}{\max_j |p_j|_v} & \text{if } v \text{ is non-Archimedean,} \end{cases} \tag{2.4.1}$$

The projective space $\mathbb{P}^r_{\mathbb{K}}$ has a standard structure of toric variety with torus $\mathbb{G}^r_{\mathrm{m},\mathbb{K}}$, included via the map $(z_1, \dots, z_r) \mapsto (1 : z_1 : \cdots : z_r)$. Thus $\overline{E}$ is a toric metrized divisor. It is a particular case of the weighted ℓ^p-metrized divisors on toric varieties studied in [20, §5.2].

The following result summarizes the basic properties of this toric metrized divisor and its combinatorial data.

Proposition 2.4.1. *The toric metrized divisor $\overline{E}$ on $\mathbb{P}^r_{\mathbb{K}}$ is semipositive and generated by small sections. For $v \in \mathfrak{M}$, its v-adic metric function is given, for $\boldsymbol{u} = (u_1, \dots, u_r) \in \mathbb{R}^r$, by*

$$\psi_{\overline{E},v}(\boldsymbol{u}) = \begin{cases} -\log\Big(1 + \sum_{j=1}^{r} \mathrm{e}^{-u_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \min(0, u_1, \dots, u_r) & \text{if } v \text{ is non-Archimedean.} \end{cases} \tag{2.4.2}$$

The polytope corresponding to E is the standard simplex Δ^r of $\mathbb{R}^r$. For $v \in \mathfrak{M}$, the v-adic roof function of $\overline{E}$ is given, for $\boldsymbol{x} = (x_1, \dots, x_r) \in \Delta^r$, by

$$\vartheta_{\overline{E},v}(\boldsymbol{x}) = \begin{cases} -\sum_{j=0}^{r} x_j \log(x_j) & \text{if } v \text{ is Archimedean,} \\ 0 & \text{if } v \text{ is non-Archimedean,} \end{cases}$$

with $x_0 = 1 - \sum_{j=1}^{r} x_j$.

Proof. The distinguished rational section of the line bundle $\mathcal{O}(E)$ corresponds to the linear form $x_0 \in \mathbb{K}[x_0, \dots, x_r]$. Hence, for an Archimedean place v and a point $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{G}^r_{\mathrm{m},\mathbb{K}}(\overline{\mathbb{K}}_v)$,

$$\psi_{\overline{E},v}(\mathrm{val}_v(\boldsymbol{z})) = \log \|s_E(\boldsymbol{z})\|_v = -\log\Big(1 + \sum_{j=1}^{r} |z_j|\Big),$$

which gives the expression in (2.4.2) for this case. The non-Archimedean case is done similarly. We can easily check that these metric functions are concave. In the Archimedean case, this can be done by computing its Hessian and verifying that it is nonpositive and, in the non-Archimedean case, it is immediate from its expression. Hence, $\overline{E}$ is semipositive.

Set s_j for the global section corresponding to the linear form $x_j \in \mathbb{K}[x_0, \dots, x_r]$, $j = 0, \dots, r$. We have that $\bigcap_{j=0}^{r} |\operatorname{div}(s_j)| = \emptyset$, and so this is a set of generating global sections. It follows from the definition of the metric in (2.4.1) that these global sections are $\overline{E}$-small. Hence, $\overline{E}$ is generated by small sections.

The fact that the polytope corresponding to E is the standard simplex is classical, see for instance [34, page 27]. When v is Archimedean, the v-adic roof function can be computed similarly as the one for the Fubini-Study metric in [19, Example 2.4.3]. When v is non-Archimedean, v-adic roof function is zero, because the metric $\|\cdot\|_v$ is canonical. □

Set $r \geq 0$. Take $\boldsymbol{m} = (m_0, \dots, m_r) \in M^{r+1}$ and $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$, and consider the polytope $\Delta = \operatorname{conv}(m_0, \dots, m_r) \subset M_\mathbb{R}$. Let X be a projective toric variety over $\mathbb{K}$ given by a fan on $N_\mathbb{R}$ that is compatible with Δ. Let $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}} \colon \mathbb{T} \to \mathbb{P}^r_\mathbb{K}$ be the monomial map in (2.3.16) and set

$$D_{\boldsymbol{m}} = \operatorname{div}(\chi^{-m_0}) + \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}} E,$$

which coincides with the Cartier divisor on X corresponding to Δ. For each $v \in \mathfrak{M}$, we consider the metric on $\mathcal{O}(D_{\boldsymbol{m}})^{\mathrm{an}}_v \simeq \mathcal{O}(\varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}} E)^{\mathrm{an}}_v$ defined by

$$\|\cdot\|_{\boldsymbol{m},\boldsymbol{\alpha},v} = |\alpha_0|_v^{-1} \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}} \|\cdot\|_{\overline{E},v}, \tag{2.4.3}$$

the homothecy by $|\alpha_0|_v$ of the inverse image by $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$ of the v-adic metric of $\overline{E}$. We then set

$$\overline{D}_{\boldsymbol{m},\boldsymbol{\alpha}} = (D_{\boldsymbol{m}}, (\|\cdot\|_{\boldsymbol{m},\boldsymbol{\alpha},v})_{v \in \mathfrak{M}}). \tag{2.4.4}$$

Since $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$ is an equivariant map and $\overline{E}$ is toric, this is a toric metrized divisor on X.

Proposition 2.4.2. *The toric metrized divisor $\overline{D} = \overline{D}_{\boldsymbol{m},\boldsymbol{\alpha}}$ on X is semipositive and generated by small sections. For $v \in \mathfrak{M}$, its v-adic metric is given, for $p \in \mathbb{T}(\overline{\mathbb{K}}_v)$, by*

$$\|s_D(p)\|_v = \begin{cases} \Big(\sum_{j=0}^{r} |\alpha_j \chi^{m_j}(p)|_v\Big)^{-1} & \text{if } v \text{ is Archimedean,} \\ \Big(\max_{0\le j\le r} |\alpha_j \chi^{m_j}(p)|_v\Big)^{-1} & \text{if } v \text{ is non-Archimedean.} \end{cases} \tag{2.4.5}$$

The v-adic metric function of $\overline{D}$ is given, for $u \in N_{\mathbb{R}}$, by

$$\psi_{\overline{D},v}(u) = \begin{cases} -\log\Big(\sum_{j=0}^{r} |\alpha_j|_v \, \mathrm{e}^{-\langle m_j, u\rangle}\Big) & \text{if } v \text{ is Archimedean,} \\ \min_{0\le j\le r} \langle m_j, u\rangle - \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean,} \end{cases} \tag{2.4.6}$$

and the v-adic roof function of $\overline{D}$ is given, for $x \in \Delta$, by

$$\vartheta_{\overline{D},v}(x) = \begin{cases} \max_{\boldsymbol{\lambda}} \sum_{j=0}^{r} \lambda_j \log\Big(\frac{|\alpha_j|_v}{\lambda_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \max_{\boldsymbol{\lambda}} \sum_{j=0}^{r} \lambda_j \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean,} \end{cases} \tag{2.4.7}$$

the maximum being over the vectors $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_r) \in \mathbb{R}_{\ge 0}^{r+1}$ with $\sum_{j=0}^{r} \lambda_j = 1$ such that $\sum_{j=0}^{r} \lambda_j m_j = x$.

Proof. Set $\overline{D'} = \varphi_{\boldsymbol{m},\boldsymbol{\alpha}}^{*}\overline{E}$ for short. This is a toric metrized divisor on X that is semipositive and generated by small sections, due to Proposition 2.4.1 and the preservation of these properties under inverse image. Since the v-adic metrics of $\overline{D}$ are homothecies of those of $\overline{D'}$, it follows that $\overline{D}$ is semipositive too. Moreover, a global section ς of $\mathcal{O}(D') \simeq \mathcal{O}(D)$ is $\overline{D'}$-small if and only if the global section $\alpha_0\,\varsigma$ is $\overline{D}$-small. It follows that $\overline{D}$ is also generated by small sections.

Using (2.4.1) and the definition of the monomial map $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$, for $v \in \mathfrak{M}$, the v-adic metric of $\overline{D'}$ is given, for $p \in \mathbb{T}(\overline{\mathbb{K}}_v)$, by

$$\|s_{D'}(p)\|_v = \begin{cases} \bigg(\sum_{j=0}^{r} \Big|\frac{\alpha_j}{\alpha_0}\chi^{m_j - m_0}(p)\Big|_v\bigg)^{-1} & \text{if } v \text{ is Archimedean,} \\ \bigg(\max_{0\le j\le r} \Big|\frac{\alpha_j}{\alpha_0}\chi^{m_j - m_0}(p)\Big|_v\bigg)^{-1} & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Since $D = \operatorname{div}(\chi^{-m_0}) + D'$, their distinguished rational sections are related by $s_D = \chi^{-m_0} s_{D'}$. It follows from (2.4.3) that

$$\|s_D(p)\|_v = |\alpha_0|_v^{-1} |\chi^{-m_0}(p)|_v \, \|s_{D'}(p)\|_v,$$

which implies the formulae in (2.4.5). As a consequence, we obtain also the expressions for the v-adic metric functions of $\overline{D}$.

For its roof function, consider first the linear map $H\colon N_{\mathbb{R}} \to \mathbb{R}^{r+1}$ given, for $u \in N_{\mathbb{R}}$, by $H(u) = (\langle m_0, u\rangle, \dots, \langle m_r, u\rangle)$. For each place v, consider the concave function $g_v\colon \mathbb{R}^{r+1} \to \mathbb{R}$ given, for $\boldsymbol{\nu} \in \mathbb{R}^{r+1}$, by

$$g_v(\boldsymbol{\nu}) = \begin{cases} -\log\Big(\displaystyle\sum_{j=0}^{r} |\alpha_j|_v \,\mathrm{e}^{-\nu_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \displaystyle\min_{0\le j\le r} \nu_j - \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Notice that $\psi_{\overline{D},v} = H^* g_v$. The domain of the Legendre-Fenchel dual $g_v^\vee$ of g_v is the simplex S given as the convex hull of the vectors in the standard basis of $\mathbb{R}^{r+1}$; and $g_v^\vee$ is given, for $\boldsymbol{\lambda} \in S$, by

$$g_v^\vee(\boldsymbol{\lambda}) = \begin{cases} \displaystyle\sum_{j=0}^{r} \lambda_j \log\Big(\frac{|\alpha_j|_v}{\lambda_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \displaystyle\max_{\boldsymbol{\lambda}} \sum_{j=0}^{r} \lambda_j \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

For the Archimedan case, this follows similarly to [19, Example 2.4.3], and is also proved in [20, Proposition 5.8]. For the non-Archimedean case, it follows from Example 2.3.29. Then, by [19, Proposition 2.3.8(3)], the v-adic roof function $\vartheta_{\overline{D},v}$ is the direct image under the dual map $H^\vee$ of the Legendre-Fenchel dual $g_v^\vee$, which gives the stated formulae in (2.4.7). □

Definition 2.4.3. Let $f \in \mathbb{K}[M]$ be a Laurent polynomial and X be a projective toric variety over $\mathbb{K}$ given by a fan on $N_{\mathbb{R}}$ that is compatible with the Newton polytope $\mathcal{N}(f)$. Write $f = \sum_{j=0}^{r} \alpha_j \chi^{m_j}$ with $m_j \in M$ and $\alpha_j \in \mathbb{K}^\times$. The *toric metrized divisor on X associated to f* is defined as

$$\overline{D}_f = \overline{D}_{\boldsymbol{m},\boldsymbol{\alpha}},$$

the toric metrized divisor in (2.4.4) for the data $\boldsymbol{m} = (m_0, \dots, m_r) \in M^{r+1}$ and $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$. It does not depend on the ordering of the terms of f. For $v \in \mathfrak{M}$, we denote by $\psi_{f,v}$ and $\vartheta_{f,v}$ the v-adic metric and roof functions of $\overline{D}_f$, respectively.

Lemma 2.4.4. *With notation as in Definition 2.4.3, the global section of $\mathcal{O}(D_f)$ associated to f is $\overline{D}_f$-small.*

Proof. Set $\overline{D} = \overline{D}_f$ for short, and let $s = f s_D$ be the global section of $\mathcal{O}(D)$ associated to f. For $v \in \mathfrak{M}$ and $p \in \mathbb{T}(\overline{\mathbb{K}}_v)$,

$$\|s(p)\|_v = |f(p)|_v \,\|s_D(p)\|_v = \Big|\sum_{j=0}^{r} \alpha_j \chi^{m_j}(p)\Big|_v \|s_D(p)\|_v.$$

It follows from (2.4.5) that $\|s\|_v \le 1$ on $\mathbb{T}(\overline{\mathbb{K}}_v)$, and so s is $\overline{D}$-small. □

The following result corresponds to (2.1.3) in the introduction.

Theorem 2.4.5. *Let $f_1, \dots, f_n \in \mathbb{K}[M]$, and let X be a proper toric variety with torus $\mathbb{T}_M$ and $\overline{D}_0$ a nef toric metrized divisor on X. Let $\Delta_0 \subset M_{\mathbb{R}}$ be the polytope of D_0 and, for $v \in \mathfrak{M}$, let $\vartheta_{0,v} \colon \Delta_i \to \mathbb{R}$ be v-adic roof function of $\overline{D}_0$. For $i = 1, \dots, n$, let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of f_i and, for $v \in \mathfrak{M}$, let $\vartheta_{i,v} \colon \Delta_i \to \mathbb{R}$ be the v-adic roof function on the metric associated to f_i. Then*

$$\mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n)) \le \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}(\vartheta_{0,v}, \dots, \vartheta_{n,v}).$$

Proof. Let Σ be the complete fan corresponding to the proper toric variety X. By taking a refinement, we can assume without loss of generality that Σ is regular and compatible with the Newton polytopes Δ_i, $i = 1, \dots, n$. Hence X is a projective toric variety and $\overline{D}_0$ a nef toric metrized divisor, and there are nef toric Cartier divisors D_i, $i = 1, \dots, n$, corresponding to these Newton polytopes.

For $i = 1, \dots, n$, we denote by $\overline{D}_i$ the toric metrized divisor associated to f_i (Definition 2.4.3). By Proposition 2.4.2, $\overline{D}_i$ is semipositive and generated by small sections and, by Lemma 2.4.4, the global section s_i of $\mathcal{O}(D_i)$ corresponding to f_i is $\overline{D}_i$-small. Applying Corollary 2.3.24 and Theorem 2.3.31,

$$\mathrm{h}_{\overline{D}_0}\Big(\prod_{i=1}^{n} \mathrm{div}(s_i)\Big) \le \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_n}(X) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_n,v}).$$

Due to Proposition 2.2.9(2), the inequality $Z(f_1, \dots, f_n) \le \prod_{i=1}^{n} \mathrm{div}(s_i)$ holds. By the linearity of the global height and the nefness of $\overline{D}_0$,

$$\mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n)) \le \mathrm{h}_{\overline{D}_0}\Big(\prod_{i=1}^{n} \mathrm{div}(s_i)\Big),$$

which concludes the proof. $\square$

Definition 2.4.6. Let $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r) \in (\mathbb{K}^\times)^r$ with $r \ge 1$. For $v \in \mathfrak{M}$, the *v-adic logarithmic length* of $\boldsymbol{\alpha}$ is defined as

$$\ell_v(\boldsymbol{\alpha}) = \begin{cases} \log\big(\sum_{j=0}^{r} |\alpha_j|_v\big) & \text{if } v \text{ is Archimedean,} \\ \log\big(\max_{0 \le j \le r} |\alpha_j|_v\big) & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

The *logarithmic length* of $\boldsymbol{\alpha}$ is defined as $\ell(\boldsymbol{\alpha}) = \sum_{v \in \mathfrak{M}} n_v \ell_v(\boldsymbol{\alpha})$.

For a Laurent polynomial $f \in \mathbb{K}[M]$, we define its *v-adic logarithmic length*, denoted by $\ell_v(f)$, as the v-adic length of its vector of coefficients, $v \in \mathfrak{M}$. We also define its *logarithmic length*, denoted by $\ell(f)$, as the length of its vector of coefficients.

Lemma 2.4.7. *Let $\vartheta_i\colon \Delta_i \to \mathbb{R}$ be a concave function on a convex body, $i = 0, \dots, n$. Then*

$$\mathrm{MI}_M(\vartheta_0, \dots, \vartheta_n) \leq \sum_{i=0}^{n} \Big(\max_{x\in\Delta_i} \vartheta_i(x)\Big) \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$$

Proof. Set $c_i = \max_{x\in\Delta_i} \vartheta_i(x)$ for short. By the monotonicity of the mixed integral, see [67, Proposition 8.1],

$$\mathrm{MI}_M(\vartheta_0, \dots, \vartheta_n) \leq \mathrm{MI}_M(c_0|_{\Delta_0}, \dots, c_n|_{\Delta_n}),$$

where $c_i|_{\Delta_i}$ denotes the constant function c_i on the convex body Δ_i. By [67, formula (8.3)],

$$\mathrm{MI}_M(c_0|_{\Delta_0}, \dots, c_n|_{\Delta_n}) = \sum_{i=0}^{n} c_i \,\mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n),$$

giving the stated inequality. □

The following result corresponds to the inequality (2.1.4) in the introduction.

Corollary 2.4.8. *With notation as in Theorem 2.4.5,*

$$\begin{aligned}\mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n)) \leq \Big(\sum_{v\in\mathfrak{M}} n_v \max_{x\in\Delta_0} \vartheta_{0,v}(x)\Big) &\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) \\ &+ \sum_{i=1}^{n} \ell(f_i)\,\mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n).\end{aligned}$$

In particular, for the canonical metric on D_0 (Example 2.3.26),

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(Z(f_1, \dots, f_n)) \leq \sum_{i=1}^{n} \ell(f_i)\,\mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n).$$

Proof. For $1 \leq i \leq n$ and $v \in \mathfrak{M}$, let $\vartheta_{i,v}$ be the v-adic roof function of the toric semipositive metric associated to f_i. From the definition of the Legendre-Fenchel dual, the maximum of a concave function ϑ is $-\vartheta^\vee(0)$ (see also [71, Theorem 23.5]). Using (2.4.7), we compute the values of $-\psi_{i,v}(0) = -\vartheta_{i,v}^\vee(0)$ and obtain

$$\max_{x\in\Delta_i} \vartheta_{i,v}(x) = \ell_v(f_i). \tag{2.4.8}$$

The first statement follows then from Theorem 2.4.5 and Lemma 2.4.7. The second statement is a particular case of the first one, using the fact that the v-adic roof functions of $\overline{D}_0^{\mathrm{can}}$ are the zero functions on Δ_0. □

We readily derive from the previous corollary the following version of the arithmetic Bézout theorem.

Corollary 2.4.9. *Let $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$ and let $\overline{D}^{\mathrm{can}}$ be the divisor at infinity of $\mathbb{P}^n_{\mathbb{K}}$ equipped with the canonical metric. Then*

$$\mathrm{h}_{\overline{D}^{\mathrm{can}}}(Z(f_1, \dots, f_n)) \leq \sum_{i=1}^{n} \Big(\prod_{j \neq i} \deg(f_j) \Big) \ell(f_i),$$

where deg *denotes the total degree of the corresponding polynomial.*

Proof. Notice that, for each $i = 1, \dots, n$, the Newton polytope of f_i is contained in $\deg(f_i)\Delta^n$. Then by the monotonicity and linearity of the mixed volume

$$\mathrm{MV}_{\mathbb{Z}^n}(\Delta^n, \dots, \widehat{\Delta_i}, \dots, \widehat{\Delta_n}) \leq \prod_{j \neq i} \deg(f_j)\, \mathrm{MV}_{\mathbb{Z}^n}(\Delta^n, \dots, \Delta^n) = \prod_{j \neq i} \deg(f_j),$$

where the Δ_i's are the respective Newton polytopes of the f_i's. □

2.4.2 Examples

The two families of examples have as objective to illustrate two aspects of the bounds obtained above. With the first family of examples we provide a case in which both these bounds do approach the height of the 0-cycle; while with the second one we show a situation where the bound of Theorem 2.4.5 is sharp, and that of Corollary 2.4.8 is not.

We keep the notation of §2.4.1. We need the the following auxiliary computation of mixed volumes. For its proof, we recall that the mixed volume of a family of polytopes $\Delta_i \subset \mathbb{R}^n$, $i = 1, \dots, n$, can be decomposed in terms of mixed volumes of their lower dimensional faces as

$$\mathrm{MV}_n(\Delta_1, \dots, \Delta_n) = - \sum_{u \in \mathrm{S}^{n-1}} \Psi_{\Delta_1}(u)\, \mathrm{MV}_{n-1}(\Delta_2^u, \dots, \Delta_n^u), \tag{2.4.9}$$

where S^{n-1} is the unit sphere of $\mathbb{R}^n$, Ψ_{Δ_1} is the support function of Δ_1 as in (2.2.6), Δ_i^u is the unique face of Δ_i that minimizes the functional u on this polytope, and MV_n and MV_{n-1} denote the mixed volume functions associated to the Lebesgue measure of $\mathbb{R}^n$ and $u^{\perp} \simeq \mathbb{R}^{n-1}$, respectively. In fact, this sum ranges through all the normal vectors of the facets of each polytope. We refer to [78, formula (5.1.22)] for more details.

Lemma 2.4.10. *Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope, and $m_i \in M$, $i = 2, \dots, n$, linearly independent lattice points. Denote by $\overline{0\, m_i}$ the segment between 0 and m_i, and $u \in N$ the smallest lattice point orthogonal to all the m_i's, which is unique up to a sign. Let $P = \sum_{i=2}^{n} \mathbb{Z} m_i \subset M$ be the sublattice generated by the m_i's, and P^{sat} its saturation. Then*

$$\mathrm{MV}_M(\Delta, \overline{0\, m_2}, \dots, \overline{0\, m_n}) = [P^{\mathrm{sat}} : P] \operatorname{vol}_{\mathbb{Z}} \langle \Delta, u \rangle,$$

where $\langle \Delta, u \rangle$ is the image of Δ under the functional $u \colon M_{\mathbb{R}} \to \mathbb{R}$, and $\operatorname{vol}_{\mathbb{Z}}$ represents the volume associated to the Lebesgue measure in $\mathbb{Z}$.

Proof. Choosing a basis, we identify $M = \mathbb{Z}^n$. With this identification, $\mathrm{MV}_M = \mathrm{MV}_n$, the mixed volume associated to the Lebesgue measure of $\mathbb{R}^n$. The formula in (2.4.9) applied to the polytopes $\Delta, \overline{0\, m_2}, \dots, \overline{0\, m_n}$ implies that

$$\begin{aligned}\mathrm{MV}_n(\Delta, \overline{0\, m_2}, \dots, \overline{0\, m_n}) &= -\Big(\Psi_\Delta\Big(\frac{u}{\|u\|}\Big) + \Psi_\Delta\Big(-\frac{u}{\|u\|}\Big)\Big)\,\mathrm{MV}_{n-1}(\overline{0\, m_2}, \dots, \overline{0\, m_n})\\ &= -\frac{1}{\|u\|}(\Psi_\Delta(u) + \Psi_\Delta(-u))\,\mathrm{MV}_{n-1}(\overline{0\, m_2}, \dots, \overline{0\, m_n}),\end{aligned} \tag{2.4.10}$$

where $\|u\|$ is the Euclidean norm. We have that

$$\Psi_\Delta(u) + \Psi_\Delta(-u) = \min_{x\in\Delta}\langle x, u\rangle + \min_{x\in\Delta}\langle x, -u\rangle = -\operatorname{vol}_{\mathbb{Z}}\langle \Delta, u\rangle \tag{2.4.11}$$

By the Brill-Gordan duality theorem (see for example [40, Lemma 1]), we have $\|u\| = \mathrm{vol}_{n-1}(P_\mathbb{R}/P^{\mathrm{sat}})$ where vol_{n-1} denotes the Lebesgue measure of $u^\perp$. Hence

$$\frac{1}{\|u\|}\,\mathrm{MV}_{n-1}(\overline{0\, m_2}, \dots, \overline{0\, m_n}) = \mathrm{MV}_{P^{\mathrm{sat}}}(\overline{0\, m_2}, \dots, \overline{0\, m_n}) = [P^{\mathrm{sat}} : P]. \tag{2.4.12}$$

The result follows then from (2.4.10), (2.4.11) and (2.4.12). □

Example 2.4.11. Let $d, \alpha \geq 1$ be integers and consider the system of Laurent polynomials given by

$$f_1 = x_1 - \alpha, \quad f_2 = x_2 - \alpha x_1^d, \quad \dots, \quad f_n = x_n - \alpha x_{n-1}^d \quad \in \mathbb{Q}[x_1^{\pm1}, \dots, x_n^{\pm1}].$$

Its zero set in $\mathbb{T}_{\mathbb{Z}^n} = \mathbb{G}_{\mathrm{m},\mathbb{Q}}^n$ consists of the rational point

$$p = (\alpha, \alpha^{d+1}, \dots, \alpha^{d^{n-1}+d^{n-2}+\dots+1}) \in \mathbb{T}_{\mathbb{Z}^n}(\mathbb{Q}) = (\mathbb{Q}^\times)^n.$$

Let X be a proper toric variety over $\mathbb{Q}$, and $\overline{D}_0^{\mathrm{can}}$ a nef toric Cartier divisor on X equipped with the canonical metric. Let $\Delta_0 \subset \mathbb{R}^n$ be the polytope corresponding to D_0 and, for $i = 1, \dots, n$, set

$$u_i = e_i + d e_{i+1} + \dots + d^{n-i} e_n \in \mathbb{Z}^n,$$

where the e_j's are the vectors in the standard basis of $\mathbb{Z}^n$. The height of p with respect to $\overline{D}_0^{\mathrm{can}}$ is

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(p) = \Big(\operatorname{vol}_{\mathbb{Z}}\Big\langle \Delta_0, \sum_{i=1}^n u_i\Big\rangle\Big)\log(\alpha). \tag{2.4.13}$$

To prove this, let $v \in \mathfrak{M}_\mathbb{Q}$. By (2.3.14), the local height of p with respect to the pair $(\overline{D}_0^{\mathrm{can}}, s_{D_0})$ is given by

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}},v}(p, s_{D_0}) = -\log\|s_{D_0}(p)\|_{v,\mathrm{can}} = -\Psi_{\Delta_0}\big(\mathrm{val}_v(p)\big).$$

Set $u = \sum_{i=1}^n u_i$ for short. Since $\mathrm{val}_v(p) = -\log|\alpha|_v\, u$,

$$-\Psi_{\Delta_0}\big(\mathrm{val}_v(p)\big) = \begin{cases} \log|\alpha|_v \max\limits_{m\in\Delta_0\cap\mathbb{Z}^n} \langle m,u\rangle & \text{if } v=\infty,\\ \log|\alpha|_v \min\limits_{m\in\Delta_0\cap\mathbb{Z}^n} \langle m,u\rangle & \text{if } v\neq\infty. \end{cases}$$

By adding these contributions,

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(p) = \log(\alpha)\Big(\max_{m\in\Delta_0\cap\mathbb{Z}^n}\langle m,u\rangle - \min_{m\in\Delta_0\cap\mathbb{Z}^n}\langle m,u\rangle\Big),$$

which gives the formula in (2.4.13).

Next we compare the value of the height of p with the bounds given by Corollary 2.4.8. We have $\ell(f_i) = \log(\alpha+1)$ for all i. Consider the dual basis of the u_i's, given by

$$m_1 = e_1, m_2 = e_2 - de_1, \dots, e_n - de_{n-1} \in \mathbb{Z}^n.$$

For $i=1,\dots,n$, the Newton polytope Δ_i of f_i is a translate of the segment $\overline{0\,m_i}$, and u_i is the smallest lattice point in the line $(\sum_{j\neq i}\mathbb{R}m_j)^\perp$. Moreover the sublattice $\sum_{j\neq i}\mathbb{Z}m_i$ is saturated. By Lemma 2.4.10

$$\mathrm{MV}_{\mathbb{Z}^n}(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n) = \mathrm{vol}_{\mathbb{Z}}\langle\Delta_0,u_i\rangle.$$

Therefore, the bound given by Corollary 2.4.8 is

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(p) \leq \Big(\sum_{i=1}^n \mathrm{vol}_{\mathbb{Z}}\langle\Delta_0,u_i\rangle\Big)\log(\alpha+1).$$

Example 2.1.1 in the introduction consists of the particular cases corresponding to the polytopes $\Delta_0 = \Delta^n$, the standard simplex of $\mathbb{R}^n$, and $\Delta_0 = \mathrm{conv}(0,m_1,\dots,m_n)$.

In the following example, we exhibit a situation where the difference between the bounds given by the results in §2.4.1 is noticeable. Recall that passing from Theorem 2.4.5 to Corollary 2.4.8 amounts to replacing the local roof functions by constant functions on the polytope bounding them from above. Hence, to maximize the discrepancy between these two concave functions, we look for local roof functions that are tent-shaped, which is the situation where the difference between the mean value and the maximum value of these functions is the greatest possible.

Example 2.4.12. Let $\alpha \geq 1$ be an integer, and consider the system of Laurent polynomials defined as

$$f_i = x_i - \alpha \in \mathbb{Q}[x_1^{\pm1},\dots,x_n^{\pm1}], \quad \text{for } i=1,\dots,n,$$

Its zero set in $\mathbb{G}^n_{\mathrm{m},\mathbb{Q}}$ is the rational point $p = (\alpha, \dots, \alpha) \in (\mathbb{Q}^\times)^n$. Let $X = \mathbb{P}^n_{\mathbb{Q}}$ and let $\overline{E}^{\mathrm{can}}$ be the divisor of the hyperplane at infinity equipped with the canonical metric. Then the height of p with respect to $\overline{E}^{\mathrm{can}}$ is

$$\mathrm{h}_{\overline{E}^{\mathrm{can}}}(p) = \log(\alpha).$$

Next we compare the value of this height with the bound given by Theorem 2.4.5. Since the explicit computation of the mixed integrals appearing in this bound is somewhat involved, instead of giving its exact value we are going to approximate it with an upper bound that is easier to compute.

The polytope associated to the toric Cartier divisor E is $\Delta_0 = \Delta^n$, the standard simplex of $\mathbb{R}^n$. For each $v \in \mathfrak{M}_{\mathbb{Q}}$, the v-adic roof function $\vartheta_{0,v}$ of $\overline{E}^{\mathrm{can}}$ is the zero function on this simplex.

For each $i = 1, \dots, n$, let $\Delta_i = \mathcal{N}(f_i) \subset \mathbb{R}^n$ be the Newton polytope of f_i, which coincides with the segment $\overline{0\, e_i}$. For $v \in \mathfrak{M}_{\mathbb{Q}}$, let $\vartheta_{i,v}$ be the v-adic roof function associated to f_i (Definition 2.4.3). This function is given, for $t\, e_i \in \Delta_i = \overline{0\, e_i}$, by

$$\vartheta_{i,\infty}(t\, e_i) = \begin{cases} (1-t)\log(\alpha) - t\log t - (1-t)\log(1-t) & \text{if } v = \infty, \\ (1-t)\log|\alpha|_v & \text{if } v \neq \infty. \end{cases}$$

For the Archimedean place, the v-adic roof functions are nonnegative, and so their mixed integral can be expressed as a mixed volume

$$\mathrm{MI}_{\mathbb{Z}^n}(\vartheta_{0,\infty}, \dots, \vartheta_{n,\infty}) = \mathrm{MV}_{\mathbb{Z}^{n+1}}(\widetilde{\Delta}_0, \dots, \widetilde{\Delta}_n), \tag{2.4.14}$$

with $\widetilde{\Delta}_i = \mathrm{conv}\left(\mathrm{graph}(\vartheta_{i,\infty}), \Delta_i \times \{0\}\right) \subset \mathbb{R}^n \times \mathbb{R}$. Consider the concave function $\vartheta : \Delta^n \to \mathbb{R}$ defined by

$$\boldsymbol{x} = (x_1, \dots, x_n) \longmapsto \log(2) + \log(\alpha)\Big(1 - \sum_{i=1}^n x_i\Big),$$

and set $\widetilde{\Delta} = \mathrm{conv}\left(\mathrm{graph}(\vartheta), \Delta^n \times \{0\}\right) \subset \mathbb{R}^n \times \mathbb{R}$. Notice that $\vartheta_{i,\infty} \leq \vartheta$ on Δ_i, and so $\widetilde{\Delta}_i \subset \widetilde{\Delta}$, $i = 0, \dots, n$. By the monotony of the mixed volume,

$$\begin{aligned} \mathrm{MV}_{\mathbb{Z}^{n+1}}(\widetilde{\Delta}_0, \dots, \widetilde{\Delta}_n) &\leq \mathrm{MV}_{\mathbb{Z}^{n+1}}(\widetilde{\Delta}, \dots, \widetilde{\Delta}) = (n+1)! \int_{\Delta^n} \vartheta \,\mathrm{d}\boldsymbol{x} \\ &= (n+1)!\Big(\log(2)\mathrm{vol}(\Delta^n) + \log(\alpha)\int_{\Delta^n}\Big(1 - \sum_{i=1}^n x_i\Big)\mathrm{d}\boldsymbol{x}\Big) = (n+1)\log(2) + \log(\alpha). \end{aligned} \tag{2.4.15}$$

When v is non-Archimedean, we have that $|\alpha|_v \leq 1$ because α is an integer. Hence $\vartheta_{i,v} \leq 0$, and so the mixed integral of these concave functions is nonpositive. Theorem 2.4.5 together with (2.4.14) and (2.4.15) gives the upper bound

$$\mathrm{h}_{\overline{E}^{\mathrm{can}}}(p) \leq (n+1)\log(2) + \log(\alpha).$$

To conclude the example, we compute the bound given by Corollary 2.4.8. For $i=1,\dots,n$, we have that $\ell(f_i)=\log(\alpha+1)$ and $\mathrm{MV}_{\mathbb{Z}^n}(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n)=1$. Hence, this bound reduces to

$$\mathrm{h}_{\overline{E}^{\mathrm{can}}}(p)\le n\log(\alpha+1),$$

concluding the study of this example.

2.4.3 Application to $\boldsymbol{u}$-resultants and geometric representations

Fix $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $M\simeq\mathbb{Z}^n$ a lattice. As an application of our results, we bound the size of the coefficients of the $\boldsymbol{u}$-resultant of the direct image of this cycle under an equivariant map. It corresponds to Theorem 2.1.2 in the introduction, for general adelic fields satisfying the product formula. We first introduce the notion of $\boldsymbol{u}$-resultant of a 0-cycle.

Definition 2.4.13. Let $W\in Z_0(\mathbb{P}^r_{\mathbb{K}})$ be a 0-cycle of a projective space over $\mathbb{K}$ and $\boldsymbol{u}=(u_0,\dots,u_r)$ a group of $r+1$ variables. Write $W_{\overline{\mathbb{K}}}=\sum_{\boldsymbol{q}}\mu_{\boldsymbol{q}}\,\boldsymbol{q}\in Z_0(\mathbb{P}^r_{\overline{\mathbb{K}}})$ for the 0-cycle obtained from W by the base change $\mathbb{K}\hookrightarrow\overline{\mathbb{K}}$. The *$\boldsymbol{u}$-resultant* (or *Chow form*) of W is defined as

$$\mathrm{Res}(W)=\prod_{\boldsymbol{q}}(q_0u_0+\dots+q_ru_r)^{\mu_{\boldsymbol{q}}}\in\mathbb{K}(\boldsymbol{u})^\times,$$

the product being over the points $\boldsymbol{q}=(q_0:\dots:q_r)\in\mathbb{P}^r_{\mathbb{K}}(\overline{\mathbb{K}})$ in the support of $W_{\overline{\mathbb{K}}}$. It is well-defined up to a factor in $\mathbb{K}^\times$.

The length of a Laurent polynomial (Definition 2.4.6) is invariant under adelic field extensions and multiplication by scalars. It is also submultiplicative, in the sense that it satisfies the inequality

$$\ell(fg)\le\ell(f)+\ell(g)$$

for any Laurent polynomials $f,g\in\mathbb{K}[M]$.

Theorem 2.4.14. *Let $f_1,\dots,f_n\in\mathbb{K}[M]$, $\boldsymbol{m}_0\in M^{r+1}$ and $\boldsymbol{\alpha}_0\in(\mathbb{K}^\times)^{r+1}$ with $r\ge0$. Set $\Delta_0=\mathrm{conv}(m_{0,0},\dots,m_{0,r})\subset M_{\mathbb{R}}$ and let $\varphi:\mathbb{T}_M\to\mathbb{P}^r_{\mathbb{K}}$ be the monomial map associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (2.3.16). For $i=1,\dots,n$, let $\Delta_i\subset M_{\mathbb{R}}$ be the Newton polytope of f_i, and $\boldsymbol{\alpha}_i$ the vector of nonzero coefficients of f_i. Then*

$$\ell(\mathrm{Res}(\varphi_*Z(f_1,\dots,f_n)))\le\sum_{i=0}^n\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n)\,\ell(\boldsymbol{\alpha}_i).$$

Proof. Write $Z(f_1,\dots,f_n)_{\overline{\mathbb{K}}}=\sum_p\mu_p\,p$, where the sum ranges over all points $p\in\mathbb{T}_M(\overline{\mathbb{K}})$. Since the length is invariant under adelic field extensions and submultiplicative, we deduce that

$$\ell(\mathrm{Res}(\varphi_*Z(f_1,\dots,f_n)))\le\sum_p\mu_p\,\ell\big(\alpha_{0,0}\chi^{m_{0,0}}(p)\,u_0+\dots+\alpha_{0,r}\chi^{m_{0,r}}(p)\,u_r\big).\qquad(2.4.16)$$

Let X be a proper toric variety over $\mathbb{K}$ defined by a fan that is compatible with Δ_i, $i = 0, \dots, n$, and let $\overline{D}_0$ be the toric metrized divisor on X associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (2.4.4). Given a point $p \in \mathbb{T}_M(\overline{\mathbb{K}})$, we deduce from (2.4.5) that

$$\ell(\alpha_{0,0}\chi^{m_{0,0}}(p)\, u_0 + \cdots + \alpha_{0,r}\chi^{m_{0,r}}(p)\, u_r) = \mathrm{h}_{\overline{D}_0}(p). \tag{2.4.17}$$

By Proposition 2.4.2, the toric metrized divisor is semipositive and generated by small sections. In particular, it is nef. Similarly as in (2.4.8), we also get from Proposition 2.4.2 that the v-adic roof functions of $\overline{D}_0$ satisfy $\sum_{v\in\mathfrak{M}} n_v \max \vartheta_{0,v} = \ell(\boldsymbol{\alpha}_0)$. Hence, Corollary 2.4.8 implies that

$$\sum_p \mu_p \, \mathrm{h}_{\overline{D}_0}(p) \leq \sum_{i=0}^{n} \ell(\boldsymbol{\alpha}_i)\, \mathrm{MV}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n). \tag{2.4.18}$$

The statement follows then from (2.4.16), (2.4.17) and (2.4.18). □

Given a 0-dimensional variety in $(\overline{\mathbb{K}}^{\times})^r$, a way of representing this variety in terms of a family of univariate polynomials is given by the *shape lemma*. The following is a simple instance of this result.

Lemma 2.4.15. *Assume that W is a 0-dimensional variety in $(\overline{\mathbb{K}}^{\times})^r$ defined over $\mathbb{K}$. Then, there exist polynomials $h, g_0, \dots, g_r \in \mathbb{K}[t]$ such that*

$$W = \{(g_1(t)/g_0(t), \dots, g_r(t)/g_0(t)) \in (\overline{\mathbb{K}}^{\times})^r \mid t \in \overline{\mathbb{K}},\ h(t) = 0\},$$

and $\deg(g_j) < \deg(h) \leq \#W$*, for every* $j = 0, \dots, r$.

This kind of parametrizations can be tracked back to Kronecker when he introduced parametric representations of equidimensional varieties. It has since been a vast research subject in computational algebra, and are commonly known as *rational univariate representations*, or *geometric representations* in the case of varieties of any dimension. In particular, we highlight the approaches of Giusti and Heintz [37] Rouillier [72], and Krick, Pardo and Sombra [46] for their relation to $\boldsymbol{u}$-resultants.

The usual assumption on the shape lemma is that there is some coordinate that "distinguishes points". That is, there is a projection to some coordinate such that any two distinct points of W take different values under this projection. Nevertheless one can always impose a linear separating condition. For $\boldsymbol{\lambda} \in (\mathbb{K})^r \setminus \{\mathbf{0}\}$, define the linear map $L_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \lambda_1 x_1 + \cdots + \lambda_r x_r$. Then, the polynomial

$$\mathcal{L}(\boldsymbol{\lambda}) = \prod_{\substack{\boldsymbol{x},\boldsymbol{x}'\in W \\ \boldsymbol{x}\neq\boldsymbol{x}'}} (L_{\boldsymbol{\lambda}}(\boldsymbol{x}) - L_{\boldsymbol{\lambda}}(\boldsymbol{x}'))$$

is of bounded degree. Hence, there is a linear map that separates points.

The following is a proof of the Shape lemma (Lemma 2.4.15).

Proof. Fix the embedding $(\overline{\mathbb{K}}^{\times})^r \hookrightarrow \mathbb{P}^r_{\mathbb{K}}$, $(q_1, \ldots, q_r) \mapsto (1 : q_1 : \cdots : q_r)$, and a vector $\boldsymbol{\lambda} \in (\mathbb{K}^{\times})^r$ separating points of W. We then can take the polynomials of a rational univariate representation of W to be, for $t \in \overline{\mathbb{K}}^{\times}$,

$$\begin{cases} h(t) = \operatorname{Res}(W)(1, t\,\lambda_1, \ldots, t\,\lambda_r); \\ g_j(t) = \dfrac{\partial \operatorname{Res}(W)}{\partial\, u_j}(1, t\,\lambda_1, \ldots, t\,\lambda_r), \quad \text{for } j = 0, \ldots, n. \end{cases} \tag{2.4.19}$$

Notice that, since $W \subset (\overline{\mathbb{K}}^{\times})^r$ is reduced, every $\boldsymbol{u} \in (\overline{\mathbb{K}}^{\times})^{r+1}$ such that $\operatorname{Res}(W)(\boldsymbol{u}) = 0$ determines a point $\left(\frac{\partial \operatorname{Res}(W)}{\partial\, u_0}(\boldsymbol{u}) : \cdots : \frac{\partial \operatorname{Res}(W)}{\partial\, u_r}(\boldsymbol{u})\right)$ in W. The fact that $L_{\boldsymbol{\lambda}}(\boldsymbol{x}) \neq L_{\boldsymbol{\lambda}}(\boldsymbol{x}')$ for any two distinct points $\boldsymbol{x}, \boldsymbol{x}' \in W$, implies that one can simply take $\boldsymbol{u}$ ranging through a line as in (2.4.19). □

To deal with multiplicities (and henceforth 0-cycles), one could formally codify this information in h, the multiplicity of the point in W being the one of its corresponding value t. This is however not the point of interest of geometric representations, and we continue considering varieties below.

It is our purpose to apply Theorem 2.4.14 to derive upper bounds on the logarithmic length of a such rational univariate representation of a 0-dimensional variety arising from a polynomial system.

Corollary 2.4.16. *Let $f_1, \ldots, f_n \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $\tilde{\boldsymbol{m}} \in (\mathbb{Z}^n)^r$ and $\tilde{\boldsymbol{\alpha}} \in (\mathbb{K}^{\times})^r$, with $r \geq 0$. Let*

$$\begin{aligned} \tilde{\varphi} : (\mathbb{K}^{\times})^n &\longrightarrow (\mathbb{K}^{\times})^r \\ \boldsymbol{p} &\longmapsto (\tilde{\alpha}_1\, \boldsymbol{p}^{\tilde{\boldsymbol{m}}_1}, \ldots, \tilde{\alpha}_r\, \boldsymbol{p}^{\tilde{\boldsymbol{m}}_r}). \end{aligned}$$

For $i = 1, \ldots, n$, let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of f_i, and $\boldsymbol{\alpha}_i$ the vector of nonzero coefficients of f_i. Set $\Delta_0 = \operatorname{conv}(\boldsymbol{0}, \tilde{\boldsymbol{m}}_1, \ldots, \tilde{\boldsymbol{m}}_r) \subset \mathbb{R}^n$, and $\boldsymbol{\alpha}_0 = (1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r)$.

Then there is a rational univariate representation of $\tilde{\varphi}_*(Z(f_1, \ldots, f_n))$, such that the logarithmic length of h, $g_0, \ldots, g_r$ is bounded above by

$$\ell(h) \leq \sum_{i=0}^{n} \mathrm{MV}_{\mathbb{Z}^n}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n)\, \ell(\boldsymbol{\alpha}_i) \;+\; \kappa\; \mathrm{MV}_{\mathbb{Z}^n}(\Delta_1, \ldots, \Delta_n);$$

and, for $j = 0, \ldots, r$,

$$\begin{aligned} \ell(g_j) \leq \log\big(\mathrm{MV}_{\mathbb{Z}^n}(\Delta_1, \ldots, \Delta_n)\big) \;+\; &\sum_{i=0}^{n} \mathrm{MV}_{\mathbb{Z}^n}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n)\, \ell(\boldsymbol{\alpha}_i) \\ &+ \kappa\,(\mathrm{MV}_{\mathbb{Z}^n}(\Delta_1, \ldots, \Delta_n)); \end{aligned}$$

where κ is a constant depending on the coefficients of the linear separating condition, and can always be taken $\kappa \leq \log\big(\mathrm{MV}_{\mathbb{Z}^n}(\Delta_1, \ldots, \Delta_n)\big)$. In particular, if the projection to a coordinate is already a separating condition, $\kappa = 0$.

Proof. Let $W = \varphi_*(Z(f_1,\dots,f_n))_{\overline{\mathbb{K}}}$. First, notice that for $r=1$ no separating condition is needed. For $r>1$, since $\mathcal{L}(\boldsymbol{\lambda})$ is of degree at most $\binom{\#W}{2}$, one can always choose a linear separating condition $L_{\boldsymbol{\lambda}}$ with a $\boldsymbol{\lambda} \in \mathbb{K}^r \setminus \{\mathbf{0}\}$ such that

$$\kappa = \ell(L_{\boldsymbol{\lambda}}) \le \log(\#W) \le \log\big(\mathrm{MV}_{\mathbb{Z}^n}(\Delta_1,\dots,\Delta_n)\big); \tag{2.4.20}$$

where the last inequality follows from the classical Bernštein-Kušnirenko, see Theorem2.2.10.

Set $\boldsymbol{m}_0 = (\mathbf{1}, \tilde{\boldsymbol{m}}_1,\dots,\tilde{\boldsymbol{m}}_r)$, and fix the natural embedding $\iota\colon (\mathbb{Q}^\times)^r \hookrightarrow \mathbb{P}^r_{\mathbb{Q}}$, given by $(q_1,\dots,q_r) \mapsto (1 : q_1 : \cdots : q_r)$. Then the monomial map associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (2.3.16), is $\varphi = \iota \circ \tilde{\varphi}$.

Take $h, g_0,\dots,g_r \in \mathbb{K}[t]$ as in (2.4.19), with $L_{\boldsymbol{\lambda}}$ chosen as above. Since the length is submultiplicative, and $\ell(L_{\boldsymbol{\lambda}}) = \kappa$, we have

$$\ell(h(t)) \le \sum_{\boldsymbol{q}\in W} \kappa + \ell\big(\mathrm{Res}(\varphi_*(Z(f_1,\dots,f_n)))\big).$$

By applying Theorem 2.4.14, we obtain the inequality in the statement for h.

Fix $j = 0,\dots,r$. For $t \in \overline{\mathbb{Q}}^\times$, following the notations in Definition 2.4.13 (and setting $q_0 = 1$ for every $\boldsymbol{q}$ for a compact expression), we have

$$\frac{\partial\ \mathrm{Res}(\varphi_*(Z(f_1,\dots,f_n)))}{\partial\ u_j}(u_0,\lambda_1u_1,\dots,\lambda_nu_n) = \sum_{\boldsymbol{q}'\in W} \lambda_j q'_j \prod_{\substack{\boldsymbol{q}\in W\\ \boldsymbol{q}\neq\boldsymbol{q}'}} (1+\lambda_1q_1u_1+\cdots+\lambda_rq_ru_r).$$

For every $\boldsymbol{q}'$, we have the following inequality of lengths

$$\ell\Big(\lambda_j q'_j \prod_{\substack{\boldsymbol{q}\in W\\ \boldsymbol{q}\neq\boldsymbol{q}'}} (1+\lambda_1q_1u_1+\cdots+\lambda_rq_ru_r)\Big) \le \ell\Big(\mathrm{Res}(\varphi_*(Z(f_1,\dots,f_n))(u_0,\lambda_1u_1,\dots,\lambda_ru_r)\Big).$$

Hence, we can derive

$$\ell(g_j(t)) \le \ell\Big(\sum_{\boldsymbol{q}'\in W} \mathrm{Res}(\varphi_*(Z(f_1,\dots,f_n))\Big) \le \ell\Big(\#W\ \mathrm{Res}(\varphi_*(Z(f_1,\dots,f_n))\Big).$$

By the classical Bernštein-Kušnirenko, the submultiplicity of the length, and Theorem 2.4.14, we obtain the inequality in the statement for g_j. $\square$

Bibliography

[1] I. Aliev and C. Smyth. Solving algebraic equations in roots of unity. *Forum Math.*, 24:641–665, 2012. 3, 16, 55

[2] F. Amoroso and E. Viada. Small points on subvarieties of a torus. *Duke Math. J.*, 150:407–442, 2009. 2, 4, 31, 33, 37

[3] F. Amoroso and E. Viada. Small points on rational subvarieties of tori. *Comment. Math. Helv.*, 87:355–383, 2012. 16, 27, 36

[4] D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9:1–4, 1975. English translation: Functional Anal. Appl. **9** (1975), 183–185. 7, 62

[5] F. Beukers and C. J. Smyth. Cyclotomic points on curves. In *Number theory for the millennium, I (Urbana, IL, 2000)*, pages 67–85. A K Peters, Natick, MA, 2002. 3, 4, 15, 26, 54

[6] F. A. Bogomolov. Points of finite order on abelian varieties. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(4):782–804, 973, 1980. 46

[7] F. A. Bogomolov. Sur l'algébricité des représentations l-adiques. *C. R. Acad. Sci. Paris Sér. A-B*, 290(15):A701–A703, 1980. 6, 18, 19, 46

[8] E. Bombieri and W. Gubler. *Heights in Diophantine geometry.* New Mathematical Monographs. Cambridge University Press, Cambridge, 2006. 21, 37

[9] E. Bombieri and U. Zannier. Algebraic points on subvarieties of $\mathbf{G}_m^n$. *Internat. Math. Res. Notices*, pages 333–347, 1995. 2, 14

[10] E. Bombieri and U. Zannier. Heights of algebraic points on subvarieties of abelian varieties. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(4):779–792 (1997), 1996. 5, 18

[11] J.-B. Bost, H. Gillet, and C. Soulé. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(4):903–1027, 1994. 9

[12] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 4 et 5.* Hermann, 1959. 26, 36

[13] N. Bourbaki. *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations.* Hermann, 1964. 73, 75

[14] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Hermann, 1970. 74

[15] A. Buium. Geometry of p-jets. *Duke Math. J.*, 82(2):349–367, 1996. 5, 18

[16] Yu. D. Burago and V. A. Zalgaller. *Geometric inequalities*, volume 285 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1988. 8

[17] J. I. Burgos Gil, A. Moriwaki, P. Philippon, and M. Sombra. Arithmetic positivity on toric varieties. *J. Algebraic Geom.*, 25:201–272, 2016. 63, 71

[18] J. I. Burgos Gil, P. Philippon, J. Rivera-Letelier, and M. Sombra. The distribution of Galois orbits of points of small height in toric varieties. e-print arXiv:1509.01011v1, 2015. 80

[19] J. I. Burgos Gil, P. Philippon, and M. Sombra. *Arithmetic geometry of toric varieties. Metrics, measures and heights*, volume 360 of *Astérisque.* Soc. Math. France, 2014. 9, 10, 62, 63, 71, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 90, 92

[20] J. I. Burgos Gil, P. Philippon, and M. Sombra. Successive minima of toric height functions. *Ann. Inst. Fourier (Grenoble)*, 65(5):2145–2197, 2015. 89, 92

[21] J. W. S. Cassels. *Local fields*, volume 3 of *London Math. Soc. Stud. Texts.* Cambridge Univ. Press, 1986. 72

[22] A. Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006. 80

[23] M. Chardin. Une majoration de la fonction de Hilbert et ses conséquences pour l'interpolation algébrique. *Bull. Soc. Math. France*, 117:305–318, 1989. 4, 6, 17, 19, 28, 57

[24] M. Chardin and P. Philippon. Régularité et interpolation. *J. Algebraic Geom.*, 8:471–481, 1999. 4, 6, 17, 19, 29, 57

[25] R. F. Coleman. Torsion points on curves and p-adic abelian integrals. *Ann. of Math. (2)*, 121(1):111–168, 1985. 5, 18

[26] J. H. Conway and A. J. Jones. Trigonometric Diophantine equations (On vanishing sums of roots of unity). *Acta Arith.*, 30:229–240, 1976. 2, 15, 43

[27] C. D'Andrea, T. Krick, and M. Sombra. Heights of varieties in multiprojective spaces and arithmetic Nullstellensätze. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(4):549–627 (2013), 2013. 8

[28] S. David and P. Philippon. Minorations des hauteurs normalisées des sous-variétés des tores. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(3):489–543, 1999. 2, 16

[29] M. Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970. 7

[30] R. Dvornicich and U. Zannier. On sums of roots of unity. *Monatsh. Math.*, 129(2):97–108, 2000. 3, 15

[31] J-H. Evertse. The number of solutions of linear equations in roots of unity. *Acta Arith.*, 89:45–51, 1999. 2, 14

[32] G. Faltings. Diophantine approximation on abelian varieties. *Ann. of Math. (2)*, 133(3):549–576, 1991. 9

[33] W. Fulton. *Intersection theory*, volume 2 of *Ergeb. Math. Grenzgeb. (3)*. Springer-Verlag, 1984. 65, 67, 68, 74

[34] W. Fulton. *Introduction to toric varieties*, volume 131 of *Ann. of Math. Stud.* Princeton Univ. Press, 1993. 65, 70, 90

[35] A. Galateau and C. Martínez. Torsion in subvarieties of abelian varieties. in preparation. 13

[36] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants.* Math. Theory Appl. Birkhäuser, 1994. 8, 61

[37] M. Giusti and J. Heintz. Algorithmes—disons rapides—pour la décomposition d'une variété algébrique en composantes irréductibles et équidimensionnelles. In *Effective methods in algebraic geometry (Castiglioncello, 1990)*, volume 94 of *Progr. Math.*, pages 169–194. Birkhäuser Boston, Boston, MA, 1991. 100

[38] M. Giusti, J. Heintz, K. Hägele, J. E. Morais, L. M. Pardo, and J. L. Montaña. Lower bounds for Diophantine approximations. *J. Pure Appl. Algebra*, 117/118:277–317, 1997. 8

[39] W. Gubler. Heights of subvarieties over M-fields. In *Arithmetic geometry (Cortona, 1994)*, Sympos. Math., XXXVII, pages 190–227. Cambridge Univ. Press, 1997. 73, 81

[40] D. R. Heath-Brown. Diophantine approximation with square-free numbers. *Math. Z.*, 187(3):335–344, 1984. 96

[41] M. Hindry. Points de torsion sur les sous-variétés de groupes algébriques. Séminaire de théorie des nombres de Bordeaux, 1986–1987. exposé n°18. 5, 18

[42] M. Hindry. Autour d'une conjecture de Serge Lang. *Invent. Math.*, 94:575–603, 1988. 1, 5, 18, 19, 25, 26, 52

[43] E. Hrushovski. The Manin-Mumford conjecture and the model theory of difference fields. *Ann. Pure Appl. Logic*, 112(1):43–115, 2001. 5, 18

[44] F. John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948. 4, 5, 17, 38

[45] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973. 7

[46] T. Krick, L. M. Pardo, and M. Sombra. Sharp estimates for the arithmetic Nullstellensatz. *Duke Math. J.*, 109(3):521–598, 2001. 8, 100

[47] A. G. Kušnirenko. Polyèdres de Newton et nombres de Milnor. *Invent. Math.*, 32:1–31, 1976. 7, 62

[48] S. Lang. Division points on curves. *Ann. Mat. Pura Appl. (4)*, 70:229–234, 1965. 1, 14, 19

[49] S. Lang. *Fundamentals of Diophantine geometry.* Springer-Verlag, New York, 1983. 1, 14

[50] S. Lang. *Algebra. Revised third edition*, volume 121 of *Graduate Texts in Math.* Springer-Verlag, 2002. 74

[51] H. Lange and W. Ruppert. Complete systems of addition laws on abelian varieties. *Invent. Math.*, 79(3):603–610, 1985. 55

[52] M. Laurent. Équations diophantiennes exponentielles. *Invent. Math.*, 78:299–327, 1984. 1, 14

[53] R. Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergeb. Math. Grenzgeb. (3).* Springer-Verlag, 2004. 65, 68

[54] L. Leroux. Computing the torsion points of a variety defined by lacunary polynomials. *Math. Comp.*, 81:1587–1607, 2012. 16

[55] D. Lombardo. Bounds for Serre's open image theorem for elliptic curves over number fields. *Algebra Number Theory*, 9(10):2347–2395, 2015. 47

[56] V. Maillot. *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables*, volume 80 of *Mém. Soc. Math. France*. Soc. Math. France, 2000. 9, 62

[57] H. B. Mann. On linear relations between roots of unity. *Mathematika*, 12:107–117, 1965. 1, 2, 15

[58] C. Martínez. The number of maximal torsion cosets in subvarieties of tori. *J. Reine Angew. Math. (to appear)*. 13

[59] C. Martínez and M. Sombra. An arithmetic Bernštein-Kušnirenko inequality. e-print arXiv:1609.00509, 2016. 61

[60] A. Mattuck. On symmetric products of curves. *Proc. Amer. Math. Soc.*, 13:82–87, 1962. 55

[61] B. Mazur. Arithmetic on curves. *Bull. Amer. Math. Soc. (N.S.)*, 14(2):207–259, 1986. 18

[62] W. Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, third edition, 2004. 22

[63] Yu. V. Nesterenko. Estimates for the characteristic function of a prime ideal. *Mat. Sb. (N.S.)*, 123(165)(1):11–34, 1984. 9

[64] T. Oda and K. Miyake. Almost homogeneous algebraic varieties under algebraic torus action. pages 373–381, 1975. 7

[65] P. Philippon. Critères pour l'indépendance algébrique. *Inst. Hautes Études Sci. Publ. Math.*, (64):5–52, 1986. 9

[66] P. Philippon. Sur des hauteurs alternatives. III. *J. Math. Pures Appl. (9)*, 74:345–365, 1995. 27, 35, 60

[67] P. Philippon and M. Sombra. A refinement of the Bernštein-Kušnirenko estimate. *Adv. Math.*, 218:1370–1418, 2008. 8, 61, 88, 94

[68] M. Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1):207–233, 1983. 1, 18

[69] M. Raynaud. Sous-variétés d'une variété abélienne et points de torsion. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 327–352. Birkhäuser Boston, Boston, MA, 1983. 1, 18, 19

[70] G. Rémond. Sur les sous-variétés des tores. *Compositio Math.*, 134(3):337–366, 2002. 2, 16

[71] R. T. Rockafellar. *Convex analysis.* Princeton Math. Series, vol. 28. Princeton Univ. Press, 1970. 94

[72] F. Rouillier. Solving zero-dimensional systems through the rational univariate representation. *Appl. Algebra Engrg. Comm. Comput.*, 9(5):433–461, 1999. 100

[73] W. M. Ruppert. Solving algebraic equations in roots of unity. *J. Reine Angew. Math.*, 435:119–156, 1993. 3, 15, 16

[74] P. Sarnak and S. Adams. Betti numbers of congruence groups. *Israel J. Math.*, 88:31–72, 1994. With an appendix by Ze'ev Rudnick. 3, 14, 15

[75] I. Satake. On the arithmetic of tube domains (blowing-up of the point at infinity). *Bull. Amer. Math. Soc.*, 79:1076–1094, 1973. 7

[76] H. P. Schlickewei. Equations in roots of unity. *Acta Arith.*, 76(2):99–108, 1996. 2, 14

[77] W. M. Schmidt. Heights of points on subvarieties of $\mathbf{G}_m^n$. In *Number theory (Paris, 1993–1994)*, volume 235 of *London Math. Soc. Lecture Note Ser.*, pages 157–187. Cambridge Univ. Press, Cambridge, 1996. 2, 14

[78] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia Math. Appl.* Cambridge Univ. Press, 1993. 95

[79] J. P. Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972. 46

[80] J. P. Serre. Résumé des cours au collège de france (1985–1986). *Ann. Collège France*, pages 95–99, 1986. 6, 19, 46

[81] J. P. Serre. Un critère d'indépendance pour une famille de représentations ℓ-adiques. *Comment. Math. Helv.*, 88(3):541–554, 2013. 46

[82] M. Sombra. Minimums successifs des variétés toriques projectives. *J. Reine Angew. Math.*, 586:207–233, 2005. 9, 63

[83] B. Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conf. Ser. in Math.* Amer. Math. Soc., 2002. 8, 61

[84] B. Teissier. Du théorème de l'index de Hodge aux inégalités isopérimétriques. *C. R. Acad. Sci. Paris Sér. A-B*, 288(4):A287–A289, 1979. 8

[85] J.-P. Wintenberger. Démonstration d'une conjecture de Lang dans des cas particuliers. *J. Reine Angew. Math.*, 553:1–16, 2002. 46

[86] S. Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8:187–221, 1995. 14